Correction

4BIM - EDEDP: modèles en temps discret
Mardi 19 janvier 2021 – Durée : 1 heure 30

Instructions

This form will be analyzed by optical reading, so please strictly respect the following rules:

- To answer, use rectangles provided below each question;
- Do not write anything outside the rectangle;
- Do not write anything either in the header nor in the margins;
- A pre-defined number of points has been assigned to each question for a total of 20.

All documents and calculators are allowed.

Identity

Fill in empty fields below and encode your student number beside.

Last and first name:

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Student number:

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1 Population dynamics of annual plants

We are interested here in the population dynamics of annual plants, based on their cycle (seeds / germination/ flowering / mortality) over one year. Each plant produces f seeds at flowering. A proportion s of these seeds gives a plant in the next generation.

Question 1 Let $N(t)$ be the number of plants just before the flowering at year t. Express $N(t+1)$ as a function of $N(t)$. Then, deduce $N(t)$ as a function of $N(0)$.

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$$N(t+1) = fsN(t)$$
$$N(t) = (fs)^t N(0)$$

Question 2 *Numerical application*: we want to compare two plant populations growing on two different kinds of soil. On soil 1, we have $f_1 = 1000$ and $s_1 = 0.005$; on soil 2, we have $f_2 = 500$ and $s_2 = 0.001$. Based on the previous question, discuss the time course for these two populations.

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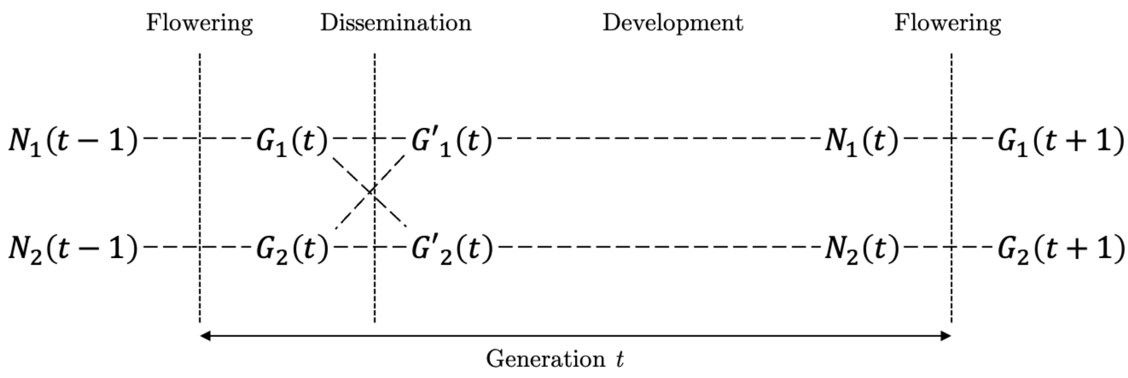
$$f_1 s_1 = 5 > 1$$
$$f_2 s_2 = 0.5 < 1$$

On soil 1, plant population will grow exponentially, while they will go to extinction on soil 2.

It was found that there was an seed exchange by dissemination between these two populations: a proportion p_1 of the seeds produced by plants on soil 1 is exported to soil 2. These plants then develop with a survival rate s_2 and produce plants with fertility f_2. In the same way, a proportion p_2 of seeds from soil 2 goes to soil 1, there having a survival rate s_1 and a fertility rate f_1. Let $N_1(t)$ be the number of plants on soil 1 just before flowering at generation t (respectively $N_2(t)$ on soil 2). Let $G_1(t)$ be the number of seeds present on soil 1 before dissemination and $G_1'(t)$ after dissemination (respectively $G_2(t)$ and $G_2'(t)$ on soil 2).



Question 3 Express $G_1'(t)$ and $G_2'(t)$ as functions of $G_1(t)$ and $G_2(t)$ and deduce matrix $\mathbf{D}$ such that $\begin{pmatrix} G_1'(t) \\ G_2'(t) \end{pmatrix} = \mathbf{D} \begin{pmatrix} G_1(t) \\ G_2(t) \end{pmatrix}$.

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$$G_1'(t) = (1 - p_1)G_1(t) + p_2 G_2(t)$$
$$G_2'(t) = p_1 G_1(t) + (1 - p_2)G_2(t)$$
$$\mathbf{D} = \begin{pmatrix} 1 - p_1 & p_2 \\ p_1 & 1 - p_2 \end{pmatrix}$$

Question 4 In the same way, determine matrices $\mathbf{F}$ and $\mathbf{S}$ such that:

$$\begin{pmatrix} G_1(t) \\ G_2(t) \end{pmatrix} = \mathbf{F} \begin{pmatrix} N_1(t-1) \\ N_2(t-1) \end{pmatrix} \text{ and } \begin{pmatrix} N_1(t) \\ N_2(t) \end{pmatrix} = \mathbf{S} \begin{pmatrix} G_1'(t)) \\ G_2'(t) \end{pmatrix}$$

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$$G_1(t) = f_1 N_1(t-1)$$

$$G_2(t) = f_2 N_2(t-1)$$

$$\mathbf{F} = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$$

$$N_1(t) = s_1 G_1'(t)$$

$$N_2(t) = s_1 G_2'(t)$$

$$\mathbf{S} = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}$$

Question 5 Deduce matrix $\mathbf{M}$ describing the whole annual cycle:

$$\begin{pmatrix} N_1(t) \\ N_2(t) \end{pmatrix} = \mathbf{M} \begin{pmatrix} N_1(t-1)) \\ N_2(t-1) \end{pmatrix}$$

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$$\mathbf{M} = \mathbf{SDF} = \begin{pmatrix} s_1 f_1 (1-p_1) & s_1 f_2 p_2 \\ s_2 f_1 p_1 & s_2 f_2 (1-p_2) \end{pmatrix}$$

Numerical application: again $f_1 = 1000$, $s_1 = 0.005$, $f_2 = 500$ and $s_2 = 0.001$.

Question 6 First case study: $p_1 = 0$ and $p_2 = 0$. Give matrix $\mathbf{M}$ and $\mathbf{M}^k$ $\forall k \in \mathbb{N}$. Deduce expressions of $N_1(t)$ and $N_2(t)$ as functions of $N_1(0)$ and $N_2(0)$, respectively. Describe and comment time course of both populations.

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$$\mathbf{M} = \begin{pmatrix} s_1 f_1 & 0 \\ 0 & s_2 f_2 \end{pmatrix}$$

$$\mathbf{M}^k = \begin{pmatrix} (s_1 f_1)^k & 0 \\ 0 & (s_2 f_2)^k \end{pmatrix}$$

$$N_1(t) = (s_1 f_1)^t N_1(0)$$

$$N_2(t) = (s_2 f_2)^t N_2(0)$$

Numerical values of parameters did not change so that we get back the same results as in question 2.

Question 7 Second case study: $p_1 = 0.5$ and $p_2 = 0$. Give matrix $\mathbf{M}$ as well as its eigenvalues. Give an eigenvector associated to the dominant eigenvalue. What happens globally for all plants over time? What are the proportions of plants in each population? Comment and conclude about the effect of the dissemination.

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$$\mathbf{M} = \begin{pmatrix} \frac{s_1 f_1}{2} & 0 \\ \frac{s_2 f_1}{2} p_1 & s_2 f_2 \end{pmatrix}$$

Eigenvalues are directly given on the diagonal (lower triangular matrix):

$$\lambda_1 = \frac{s_1 f_1}{2} = 2.5$$

$$\lambda_2 = s_2 f_2 = 0.5$$

$\lambda_1 > 1$ is the dominant eigenvalue: globally the whole plant population is growing exponentially after a sufficient long time.
A possible eigenvector associated to λ_1 is $V_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, meaning that after a sufficient long time, there are 80% of plants on soil 1, while only 20% on soil 2. Dissemination from soil 1 to soil 2 allow the plant population to persist on soil 2.

2 Molecular evolution

We are interested here in the composition in bases A and T relative to bases G and C of the genomes. For a given nucleotide sequence (read on one of the strands of the DNA molecule), we denote AT the number of bases $A + T$, and GC the number of bases $G + C$. At each generation, a proportion u of bases G and C mutates into bases A or T by substitution, and conversely, a proportion v of A and T mutates into G or C. Only mutations by substitution are considered.

Question 8 Express AT_{n+1} (the number of AT in the genome sequence at generation $n+1$) as a function of AT_n and GC_n at generation n. Same question for GC_{n+1} as a function of AT_n and GC_n.

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$$AT_{n+1} = (1-v)AT_n + uGC_n$$
$$GC_{n+1} = vAT_n + (1-u)GC_n$$

Question 9 Show that the total number of bases in the sequence is constant from one generation to the next. Comment on this result with regard to the biological hypotheses formulated above.

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$$AT_{n+1} + GC_{n+1} = (1-v)AT_n + uGC_n + vAT_n + (1-u)GC_n$$
$$AT_{n+1} + GC_{n+1} = AT_n + GC_n \forall n \in \mathbb{N}$$

Question 10 Determine matrix $\mathbf{M}$ such that $\begin{pmatrix} AT_{n+1} \\ GC_{n+1} \end{pmatrix} = \mathbf{M} \begin{pmatrix} AT_n \\ GC_n \end{pmatrix}$.

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$$\mathbf{M} = \begin{pmatrix} 1-v & u \\ v & 1-u \end{pmatrix}$$

Question 11 Calculate eigenvalues of $\mathbf{M}$ by showing that the discriminant of the characteristic equation is equal to $(u+v)^2$. Comment on the dominant eigenvalue of $\mathbf{M}$.

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The characteristic equation is given by $\lambda^2 + (u+v-2)\lambda + 1 - u - v = 0$.
The discriminant is given by $\Delta = (u+v-2)^2 - 4(1-u-v) = (u+v)^2 > 0$.
Eigenvalues are $\lambda_1 = 1-u-v$ and $\lambda_2 = 1$. As u and v are proportions, they $\in]0;1[$.
Hence $1-u-v \in]-1;1[$. The dominant eigenvalue if then $\lambda_2 = 1$ what means that the sequence length remains constant over time. This is consistent with question 9 as we do not consider neither insertions nor deletions.

Question 12 Which eigenvalue of $\mathbf{M}$ is associated with eigenvector $\begin{pmatrix} 1 \\ \frac{v}{u} \end{pmatrix}$?

Same question with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$?

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$$\mathbf{M}\begin{pmatrix} 1 \\ \frac{v}{u} \end{pmatrix} = \begin{pmatrix} 1-v & u \\ v & 1-u \end{pmatrix}\begin{pmatrix} 1 \\ \frac{v}{u} \end{pmatrix} = \begin{pmatrix} 1-v+v \\ v+\frac{v}{u}-v \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{v}{u} \end{pmatrix}$$

Hence, vector $\begin{pmatrix} 1 \\ \frac{v}{u} \end{pmatrix}$ is associated to $\lambda_2 = 1$.

$$\mathbf{M}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-v & u \\ v & 1-u \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-u-v \\ -1+u+v \end{pmatrix} = (1-u-v)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Hence, vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is associated to $\lambda_1 = 1-u-v$.

Question 13 Give the expression of matrix $\mathbf{D}$, the diagonal matrix similar to $\mathbf{M}$. Determine $\mathbf{D}^*$, the limit matrix of $\mathbf{D}^n$ when n tends towards $+\infty$. Then, deduce $\mathbf{M}^*$, the limit matrix of $\mathbf{M}^n$ when n tends towards $+\infty$.

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$$\mathbf{D} = \begin{pmatrix} 1-u-v & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{D}^n = \begin{pmatrix} (1-u-v)^n & 0 \\ 0 & 1 \end{pmatrix}$$

Because $-1 < 1-u-v < 1$, then $\mathbf{D}^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\mathbf{M} = \mathbf{PDP^{-1}} \Rightarrow \mathbf{M}^* = \mathbf{PD^*P^{-1}}$$

$$\mathbf{P^{-1}} = \frac{u}{u+v} \begin{pmatrix} \frac{v}{u} & -1 \\ 1 & 1 \end{pmatrix}$$

$$\mathbf{M}^* = \begin{pmatrix} \frac{u}{u+v} & \frac{u}{u+v} \\ \frac{v}{u+v} & \frac{v}{u+v} \end{pmatrix} = \frac{1}{u+v} \begin{pmatrix} u & u \\ v & v \end{pmatrix}$$

Question 14 For an initial composition $\begin{pmatrix} AT_0 \\ GC_0 \end{pmatrix}$, give the limit values AT^* and GC^* towards which AT_n and GC_n tend when $n \to +\infty$. Show that the $\frac{AT_n}{GC_n}$ ratio tends towards an equilibrium independent on the initial composition. Is the value of this ratio consistent with question 12?

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$$\begin{pmatrix} AT^* \\ GC^* \end{pmatrix} = \mathbf{M}^* \begin{pmatrix} AT_0 \\ GC_0 \end{pmatrix} = \frac{1}{u+v} \begin{pmatrix} u & u \\ v & v \end{pmatrix} \begin{pmatrix} AT_0 \\ GC_0 \end{pmatrix} = \frac{1}{u+v} \begin{pmatrix} u(AT_0 + GC_0) \\ v(AT_0 + GC_0) \end{pmatrix}$$

$$AT^* = \frac{u}{u+v}(AT_0 + GC_0)$$

$$GC^* = \frac{v}{u+v}(AT_0 + GC_0)$$

$$\frac{AT^*}{GC^*} = \frac{u}{v}$$

This is consistent with question 12, regarding coordinates of the eigenvector $\mathbf{V_1} = \begin{pmatrix} 1 \\ \frac{v}{u} \end{pmatrix}$ associated to the dominant eigenvalue $\lambda_2 = 1$ leading to the same proportion between At and GC at the asymptotic regime.