4BIM - EDEDP: modèles en temps discret
Mardi 19 janvier 2021 – Durée : 1 heure 30

Instructions

This form will be analyzed by optical reading, so please strictly respect the following rules:

- To answer, use rectangles provided below each question;
- Do not write anything outside the rectangle;
- Do not write anything either in the header nor in the margins;
- A pre-defined number of points has been assigned to each question for a total of 20.

All documents and calculators are allowed.

Identity

Fill in empty fields below and encode your student number beside.

Last and first name:

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Student number:

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1 Population dynamics of annual plants

We are interested here in the population dynamics of annual plants, based on their cycle (seeds / germination/ flowering / mortality) over one year. Each plant produces f seeds at flowering. A proportion s of these seeds gives a plant in the next generation.

Question 1 Let $N(t)$ be the number of plants just before the flowering at year t. Express $N(t+1)$ as a function of $N(t)$. Then, deduce $N(t)$ as a function of $N(0)$.

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Question 2 *Numerical application*: we want to compare two plant populations growing on two different kinds of soil. On soil 1, we have $f_1 = 1000$ and $s_1 = 0.005$; on soil 2, we have $f_2 = 500$ and $s_2 = 0.001$. Based on the previous question, discuss the time course for these two populations.

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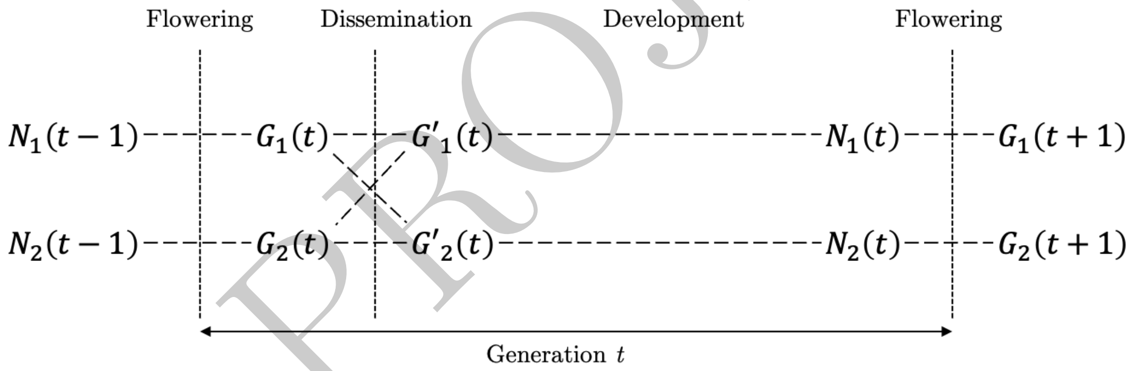
It was found that there was an seed exchange by dissemination between these two populations: a proportion p_1 of the seeds produced by plants on soil 1 is exported to soil 2. These plants then develop with a survival rate s_2 and produce plants with fertility f_2. In the same way, a proportion p_2 of seeds from soil 2 goes to soil 1, there having a survival rate s_1 and a fertility rate f_1. Let $N_1(t)$ be the number of plants on soil 1 just before flowering at generation t (respectively $N_2(t)$ on soil 2). Let $G_1(t)$ be the number of seeds present on soil 1 before dissemination and $G_1'(t)$ after dissemination (respectively $G_2(t)$ and $G_2'(t)$ on soil 2).



Question 3 Express $G_1'(t)$ and $G_2'(t)$ as functions of $G_1(t)$ and $G_2(t)$ and deduce matrix $\mathbf{D}$ such that $\begin{pmatrix} G_1'(t) \\ G_2'(t) \end{pmatrix} = \mathbf{D} \begin{pmatrix} G_1(t) \\ G_2(t) \end{pmatrix}$.

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Question 4 In the same way, determine matrices **F** and **S** such that:

$$\begin{pmatrix} G_1(t) \\ G_2(t) \end{pmatrix} = \mathbf{F} \begin{pmatrix} N_1(t-1) \\ N_2(t-1) \end{pmatrix} \text{ and } \begin{pmatrix} N_1(t) \\ N_2(t) \end{pmatrix} = \mathbf{S} \begin{pmatrix} G_1'(t)) \\ G_2'(t) \end{pmatrix}$$

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Question 5 Deduce matrix **M** describing the whole annual cycle:

$$\begin{pmatrix} N_1(t) \\ N_2(t) \end{pmatrix} = \mathbf{M} \begin{pmatrix} N_1(t-1)) \\ N_2(t-1) \end{pmatrix}$$

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Numerical application: again $f_1 = 1000$, $s_1 = 0.005$, $f_2 = 500$ and $s_2 = 0.001$.

Question 6 First case study: $p_1 = 0$ and $p_2 = 0$. Give matrix **M** and $\mathbf{M}^k$ $\forall k \in \mathbb{N}$. Deduce expressions of $N_1(t)$ and $N_2(t)$ as functions of $N_1(0)$ and $N_2(0)$, respectively. Describe and comment time course of both populations.

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Question 7 Second case study: $p_1 = 0.5$ and $p_2 = 0$. Give matrix $\mathbf{M}$ as well as its eigenvalues. Give an eigenvector associated to the dominant eigenvalue. What happens globally for all plants over time? What are the proportions of plants in each population? Comment and conclude about the effect of the dissemination.

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2 Molecular evolution

We are interested here in the composition in bases A and T relative to bases G and C of the genomes. For a given nucleotide sequence (read on one of the strands of the DNA molecule), we denote AT the number of bases $A+T$, and GC the number of bases $G+C$. At each generation, a proportion u of bases G and C mutates into bases A or T by substitution, and conversely, a proportion v of A and T mutates into G or C. Only mutations by substitution are considered.

Question 8 Express AT_{n+1} (the number of AT in the genome sequence at generation $n+1$) as a function of AT_n and GC_n at generation n. Same question for GC_{n+1} as a function of AT_n and GC_n.

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Question 9 Show that the total number of bases in the sequence is constant from one generation to the next. Comment on this result with regard to the biological hypotheses formulated above.

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Question 10 Determine matrix $\mathbf{M}$ such that $\begin{pmatrix} AT_{n+1} \\ GC_{n+1} \end{pmatrix} = \mathbf{M} \begin{pmatrix} AT_n \\ GC_n \end{pmatrix}$.

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Question 11 Calculate eigenvalues of $\mathbf{M}$ by showing that the discriminant of the characteristic equation is equal to $(u+v)^2$. Comment on the dominant eigenvalue of $\mathbf{M}$.

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Question 12 Which eigenvalue of $\mathbf{M}$ is associated with eigenvector $\begin{pmatrix} 1 \\ \frac{v}{u} \end{pmatrix}$?

Same question with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$?

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Question 13 Give the expression of matrix $\mathbf{D}$, the diagonal matrix similar to $\mathbf{M}$. Determine $\mathbf{D}^*$, the limit matrix of $\mathbf{D}^n$ when n tends towards $+\infty$. Then, deduce $\mathbf{M}^*$, the limit matrix of $\mathbf{M}^n$ when n tends towards $+\infty$.

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Question 14 For an initial composition $\begin{pmatrix} AT_0 \\ GC_0 \end{pmatrix}$, give the limit values AT^* and GC^* towards which AT_n and GC_n tend when $n \to +\infty$. Show that the $\frac{AT_n}{GC_n}$ ratio tends towards an equilibrium independent on the initial composition. Is the value of this ratio consistent with question 12?

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