INSA 4BIM 2021-2022

Difference Equations and Modelling

January the 3rd of 2022 - Duration: 1h30

Instructions

This form will be analyzed by optical reading, so please strictly respect the following rules:

- To answer, use rectangles provided below each question;
- Do not write anything outside the rectangles;
- A pre-defined number of points has been assigned to each question for a total of **23**.

All calculators are allowed, as well as an A4 sheet of paper, recto-verso, manually written and original.

Identity

Using your student card, fill in the fields below and encode your student number on the right side.

First and last names:

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Sutdent number:

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☐1	☐1	☐1	☐1	☐1	☐1	☐1
☐2	☐2	☐2	☐2	☐2	☐2	☐2
☐3	☐3	☐3	☐3	☐3	☐3	☐3
☐4	☐4	☐4	☐4	☐4	☐4	☐4
☐5	☐5	☐5	☐5	☐5	☐5	☐5
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1 Dynamics of a plant-herbivore model (16 points)

To study the interaction between certain plants and herbivores, Kang et al. (2008) proposed a two-dimensional discrete-time model utilizing leaf and herbivore biomasses as state variables[1]. The parameter space consists of the growth rate of the host population (denoted r) and a parameter describing the damage inflicted by herbivores (denoted a). Variable $t > 0$ stands for time hereafter.

Below is the dimensionless dynamical model after normalization:

$$\begin{cases} x_{t+1} = x_t e^{r(1-x_t)-ay_t} \\ y_{t+1} = x_t e^{r(1-x_t)} \left(1 - e^{-ay_t}\right) \end{cases}$$

All parameters are strictly positive. State variables x_t and y_t have no unit.

Question 1 What happens if $x_0 = 0$ at $t = 0$?

[1] Kang, Y., Armbruster, D., & Kuang, Y. (2008). Dynamics of a plant-herbivore model. *Journal of Biological Dynamics*, 2(2), 89101.

☐0 ☐0.5

Question 2 Prove that, if $x_0 > 0$ and $y_0 > 0$, then $x_t > 0$ and $y_t > 0$ $\forall t$.

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Question 3 Prove that $(0, 0)$ is a fixed point. Establish a linear relationship between x^* and y^* for all other fixed points. Among these fixed points, identify a limit case with $x_1^* > 0$ and $y_1^* = 0$. Specify the corresponding value of x_1^*.

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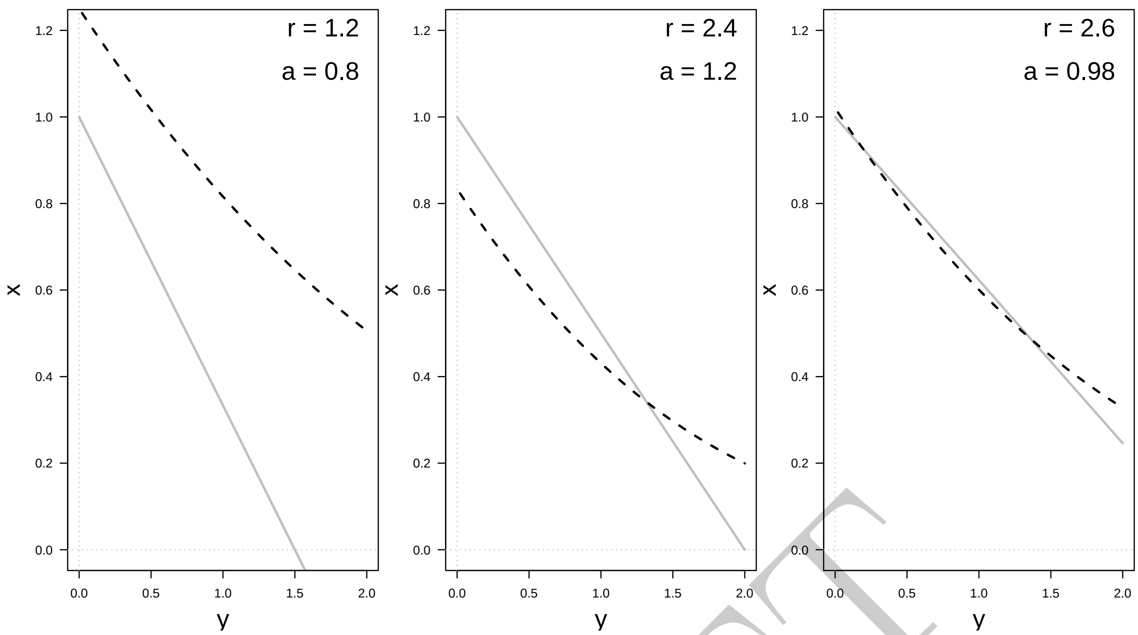
We hereafter consider the phase plane (y, x). Let f_1 and f_2 be the real functions defined by:

$$f_1(y) = 1 - \frac{a}{r}y \quad \text{and} \quad f_2(y) = \frac{y}{e^{ay} - 1}$$



Question 4 On the graphs above, identify the colour and/or the type of line associated with the function f_1. Justify your answer

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Question 5 What does both curves of f_1 and f_2 become when $r = 2$ and $a = 1$?

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Question 6 What is $\lim_{y \to 0^+} f_2(y)$? Let's recall that $e^{ay} \underset{y \to 0}{\sim} 1 + ay$.

☐0 ☐0.5 ☐1

Question 7 Finally, how many fixed points may this dynamical plant-herbivore model have?

☐0 ☐0.5 ☐1

Let's go back to the original system, that is in the plane (x, y):

$$\begin{cases} x_{t+1} = x_t e^{r(1-x_t) - a y_t} \\ y_{t+1} = x_t e^{r(1-x_t)} \left(1 - e^{-a y_t}\right) \end{cases}$$

Question 8 Give the associated Jacobian matrix $\mathbf{A}$.

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Question 9 Calculate the Jacobian matrix $\mathbf{A}_{(\mathbf{0},\mathbf{0})}$ at point $(x^*, y^*) = (0, 0)$. What is the stability of $(0, 0)$? What can you deduce from this result from a biological point of view?

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Question 10 Calculate the Jacobian matrix $\mathbf{A}_{(\mathbf{1},\mathbf{0})}$ at point $(x, y) = (1, 0)$. Under which conditions on parameters r and a is $(1, 0)$ asymptotically stable?

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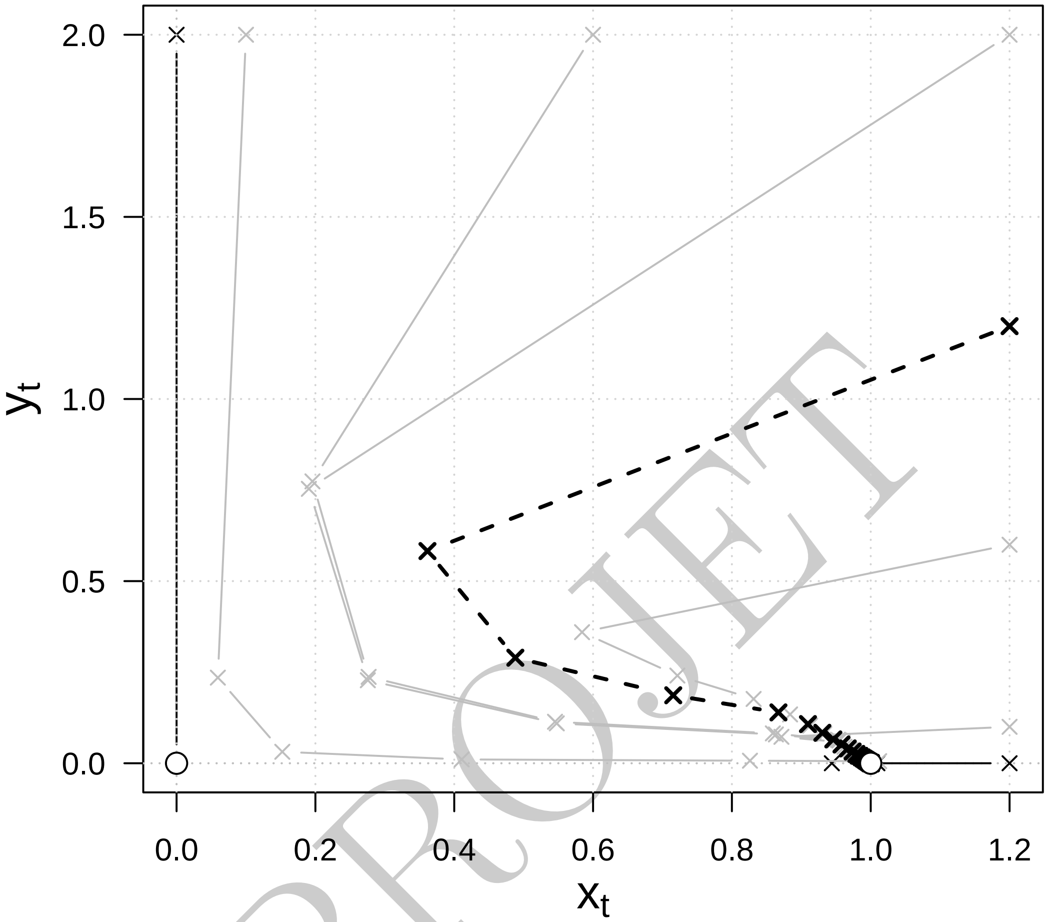
Below is the phase portrait of the dynamical plant-herbivore model with some trajectories in the phase plane (x, y).



Question 11 What is the long-term future of both plant and herbivore populations ?

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Question 12 What is the particularity of the trajectory starting at point $(1.2, 0)$?

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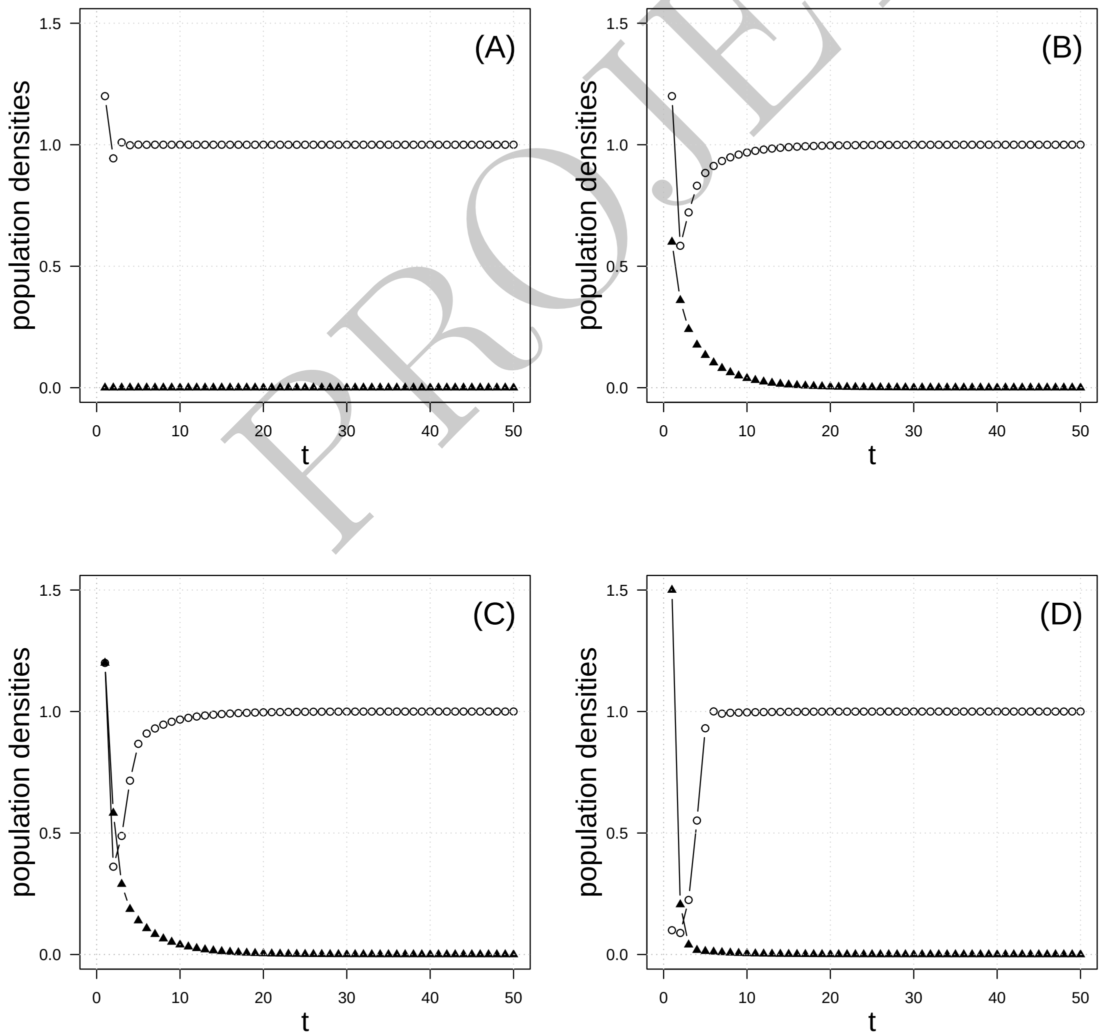
Below are some chronicle graphs.



Question 13 Which of the chronicle graphs corresponds to the dotted black trajectory on the phase portrait? Justify your answer.

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2 Matrix population model for *Leptogorgia virgulata* (7 points)

Nicholas Gotelli proposed a time-invariant matrix population model to analyse the demography of *Leptogorgia virgulata*, a shallow-water gorgonian[2] living in the northeaster Gulf of Mexico[3]. He found that the average mortality and the recruitment rates were nearly balanced, so that the population growth rate, $\ln \lambda$, was close to 0.0. An elasticity analysis showed that the recruitment contributed $< 5\%$ to the measured rate of population growth. The most important component of population growth rate was survivorship, particularly of the large size classes.

Below is the life table estimated by Gotelli:

class	survival	fecundity	growth
1	0.62	0	0.16
2	0.78	0	0.1
3	0.84	0.19	0.12
4	0.86	0.19	0.11
5	0.98	0.19	0

On the basis of the R command lines and the associated outputs given at the end of this document, answer the following questions.

Question 14 Considering these monthly vital rates of recruitment, colony growth, and mortality, write the transition matrix **M**.

[2] Une gorgone des eaux peu profondes.
[3] Gotelli, N. J. (1991). Demographic models for *Leptogorgia virgulata*, a shallow-water gorgorian. *Ecology*, 72, 457467.

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Question 15 What does the bottom-right numerical value of this matrix mean?

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Question 16 Draw the life cycle graph corresponding to **M**.

☐0 ☐0.5 ☐1

Question 17 What is the final proportion of class 1 in the total population? Justify your answer.

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Question 18 What is the numerical value of the population growth rate? Justify your answer.

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Question 19 How do you interpret the R sub-object called `$repro.value`?

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Question 20 Given the elasticity matrix, which of the vital rates mostly influences the population growth rate? What happens if this parameter decreases of 5%

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Question 21 How do you interpret the R_0 value?

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Question 22 What does the numerical output at the very end of the R script mean?

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Question 23 Can you finally confirm Gotelli's findings? Justify your answer.

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R script

Matrix population model for *Leptogorgia virgulata*

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Load-libraries

```
library("popbio")
library("diagram")
```

Vital rates

```
vital.rates <- as.matrix(data.frame(survival = c(0.62, 0.78, 0.84, 0.86, 0.98),
                                    fecundity = c(0, 0, 0.19, 0.1[illegible], 0.19),
                                    growth = c(0.16, 0.1, 0.[illegible]2, 0.1[illegible], 0)))
```

Matrix

```
M <- matrix(0, ncol = 5, nrow = 5)
M[1,] <- t(vital.rates[,"fecundity"])
diag(M) <- vital.rates[,"survival"]
M[2,1] <- vital.rates[1,"growth"]
M[3,2] <- vital.rates[2,"growth"]
M[4,3] <- vital.rates[3,"growth"]
M[5,4] <- vital.rates[4,"growth"]
```

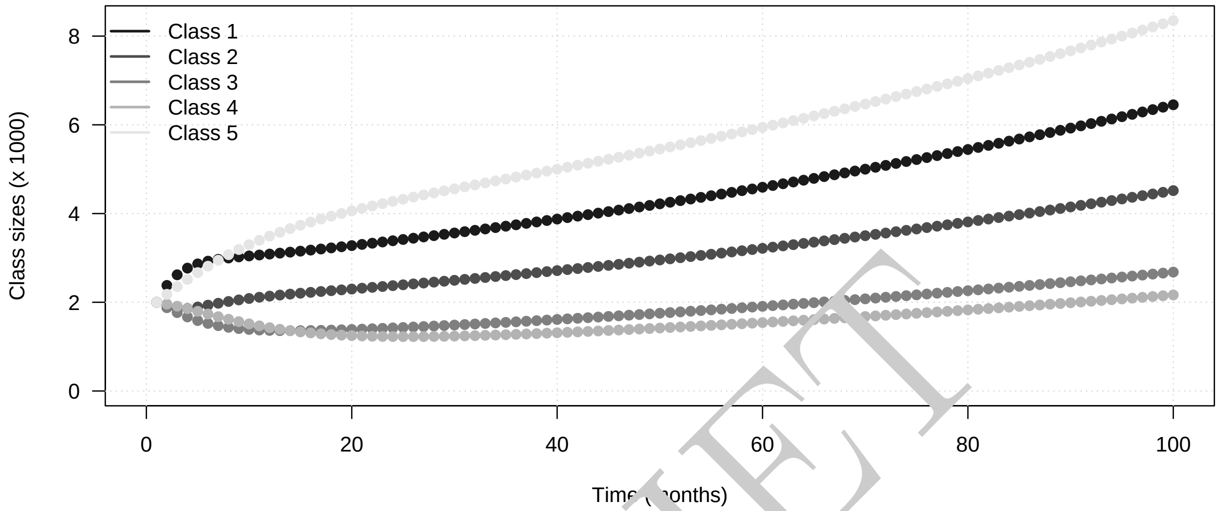
Simulations over time

```
init <- rep(2000, nrow(M))
niter <- 100
pop <- pop.projection(A = M, n = init, iterations = niter)
# pop$lambda
```

Population dynamics of each class

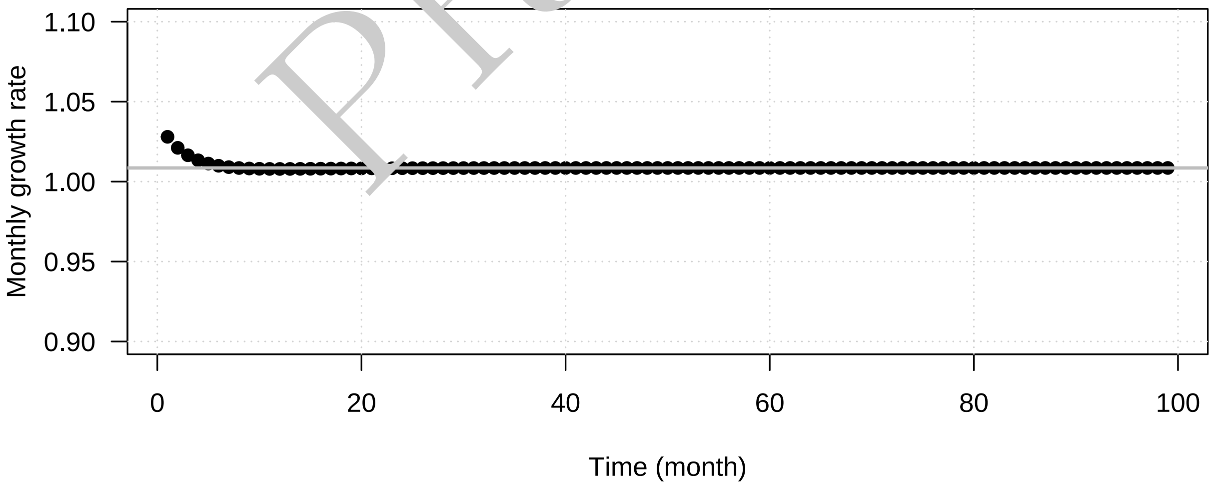
```
scale <- 1000
popdyn <- t(pop$stage.vectors)/scale
par(mfrow = c(1, 1))
colors <- c("gray10", "gray30", "gray50", "gray70", "gray90")
# rainbow(n = 5, start = 0.1, end = 0.9)
plot(0, 0, pch = NA, xlim = c(0, niter), ylim = c(0, max(popdyn)),
     xlab = "Time (months)", ylab = "Class sizes (x 1000)", las = 1)
grid()
```

```
for(i in 1: nrow(M)){
  points(popdyn[,i], type = "b", pch =19, col = colors[i])
}
legend("topleft", bty = "n", pch = NA, lty = 1, lwd = 2,
       legend = c("Class 1", "Class 2", "Class 3", "Class 4", "Class 5"),
       col = colors)
```



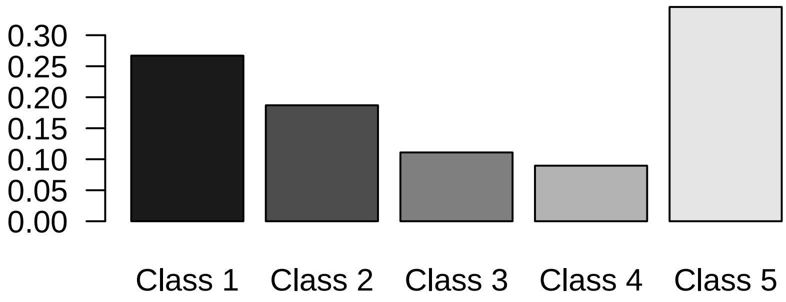
Growth rate dynamics

```
plot(1:(niter-1), pop$pop.changes, las = 1,
     pch = 19, type = "b", ylim = c(0.9, 1.1),
     xlab = "Time (month)", ylab = "Monthly growth rate")
grid()
abline(h = pop$lambda, col = "grey", lwd = 2)
```



Stable age structure

```
barplot(pop$stable.stage, las = 1, col = colors,
        names.arg = c("Class 1", "Class 2", "Class 3", "Class 4", "Class 5"))
```



Demographic parameters

```r
eigen.analysis(A = M)
```

```
$lambda1
[1] 1.008533

$stable.stage
[1] 0.26702070 0.18694599 0.11092551 0.08961691 0.34549089

$sensitivities
           [,1]       [,2]       [,3]       [,4]       [,5]
[1,] 0.06367082 0.04457709 0.02645008 0.02136907 0.08238196
[2,] 0.15461382 0.10824791 0.06422954 0.05189116 0.20005065
[3,] 0.35334346 0.24738211 0.14678564 0.11858837 0.45718158
[4,] 0.39543787 0.27685317 0.16427246 0.13271601 0.51164641
[5,] 0.42398256 0.29683781 0.17613047 0.14229612 0.54857961

$elasticities
           [,1]       [,2]        [,3]        [,4]       [,5]
[1,] 0.03914192 0.00000000 0.004982996 0.004025771 0.01552014
[2,] 0.02452891 0.08371901 0.000000000 0.000000000 0.00000000
[3,] 0.00000000 0.02452891 0.122256730 0.000000000 0.00000000
[4,] 0.00000000 0.00000000 0.019545912 0.113170101 0.00000000
[5,] 0.00000000 0.00000000 0.000000000 0.015520141 0.53305947

$repro.value
[1] 1.000000 2.428331 5.549535 6.210660 6.658977

$damping.ratio
[1] 1.144496
```

R_0

```r
net.reproductive.rate(A = M)
```

```
[1] 1.493506
```

Generation time

```r
generation.time(A = M)
```

```
[1] 47.20964
```