INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

Biomathematics 2

Bifurcation in $\mathbb{R}$

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Fish harvesting management

Consider the fish population growth logistic model:

$$x'(t) = r\,x(t)\left(1 - \frac{x}{K}\right) - Ex(t) \;\; t \in I \subset \mathbb{R},$$

with r and K positive fixed constants and E a positive varying parameter. E corresponds to the harvesting management of the fish population (red tuna, anchovies or cod for instance).

- Determine the equilibria depending on E.
- Determine their stability.
- Denoting $f_E(x) = r\,x\left(1 - \frac{x}{K}\right)$, the model is then written as $x' = f_E(x)$.

Fish harvesting management

Equilibria:
an equilibrium x^* satisfies $x'^* = 0$ that is $f_E(x^*) = 0$ or

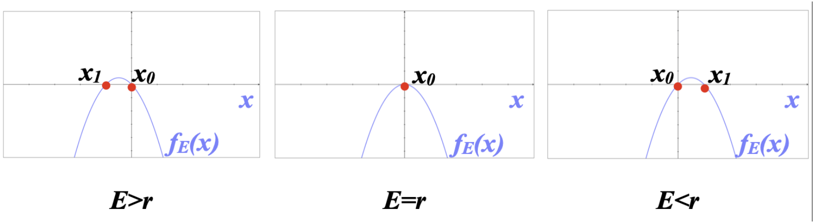
$$r x^* \left(1 - \frac{x^*}{K}\right) - E x^* = 0 \text{ which is equivalent to } x^* \left(r(1 - \frac{x^*}{K}) - E\right) = 0$$

- it is easy to see that $x^* = 0$ is always an equilibrium regardless of the values of E.
- the other equilibrium is $x^* = K\left(1 - \frac{E}{r}\right)$ given that $1 - \frac{E}{r} > 0$.
- Denote $x_0^* = 0$ and $x_1^* = K\left(1 - \frac{E}{r}\right)$.

Fish harvesting management

Stability:

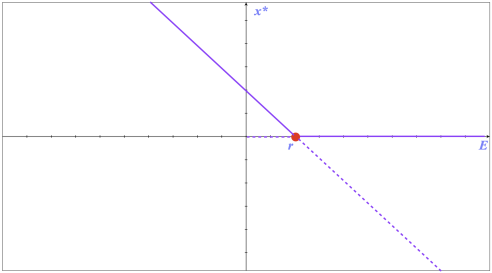
$x \mapsto f_E(x)$ is easily represented as a parabola with x_0^* always a root and x_1^* changing its value as E varies.



- if $E > r$ then $x_1^* < 0$ (not realistic), then x_1^* is unstable and x_0^* is LAS,
- if $E = r$ then $x_1^* = x_0^* = 0$ then $x_1^* = x_0^*$ is a negative shunt.
- if $E < r$ then $x_1^* > 0$, then x_1^* is LAS and x_0^* is unstable,

Fish harvesting management

Summary in one figure:

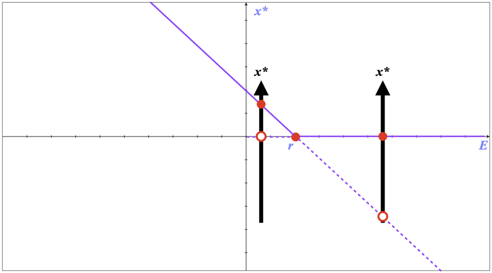


Fish harvesting management

Summary in one figure:

If E is too large, that is too much harvesting, $x_0^* = 0$ is LAS and thus the population will extinct,

If E is below r, that is reasonable harvesting, x_1^* is LAS and thus the population will survive.



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Reminder: autonomous ode

Consider the autonomous ode of order one

$$x'(t) = f(x(t)),\ t \in I \subset \mathbb{R},$$

with f satisfying the conditions of the Cauchy-Lipschitz theorem (existence and uniqueness). For instance, $f \in C^1(\mathbb{R})$ is sufficient.

- Remember that solutions x of the autonomous ode are monotonous.
- Remember that solutions of these ode can not overlap (consequence of the Cauchy-Lipschtiz).

Asymptotic behavior of the solutions of such equations is thus quite framed.

Bifurcation in $\mathbb{R}$ with one parameter: why?

The main interest resides then in inserting a varying parameter $c \in \mathbb{R}$ (*this is much closer to biological problems where parameters may vary*) such that we get a new range of variety in the sense that new equilibria can appear, or some disappear, and stability may also change. This is what we call the **bifurcation theory**. A bifurcation is crossed when the phase portrait changes.

The basic mathematical techniques you will need are available from the following web site (menu "COURS", then "Théorie des systèmes dynamiques"): http://bmm.univ-lyon1.fr/

Bifurcation in $\mathbb{R}$ with one parameter: how?

Consider now the autonomous ode of order one

$$x' = f_c(x), \text{ and } c \in \mathbb{R},$$

with $f \in C^1(\mathbb{R})$ (for simplification we note $x = x(t)$).

The objective is now to explore the number of equilibria and their stability when c changes along $\mathbb{R}$: c is then called the **bifurcation parameter**.
A good example was the Allee effect with exploitation (seen in Biomathematics 1 in Fall). Exploitation E could be the bifurcation parameter.

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Detailed content

Saddle-node bifurcation

Consider the following equation

$$x' = x^2 + c \text{ and } c \in \mathbb{R},$$

with $f_c : x \mapsto x^2 + c \in C^1(\mathbb{R})$.

Saddle-node bifurcation

Equilibria:

an equilibrium x^* satisfies $f_c(x^*) = 0$. That is

$$x^{*2} + c = 0 \text{ which is equivalent to } x^{*2} = -c.$$

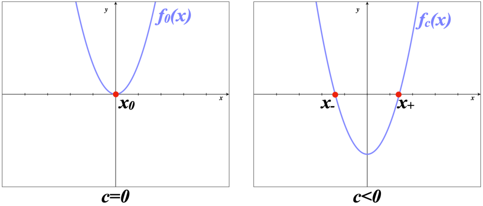
Three cases appear:

- if $c > 0$: there is no solution, so no equilibrium,
- if $c = 0$: there is one equilibrium denoted $x_0^* = 0$,
- if $c < 0$: there are two equilibria $x_-^* = -\sqrt{-c}$ and $x_+^* = \sqrt{-c}$.

Saddle-node bifurcation

Stability:

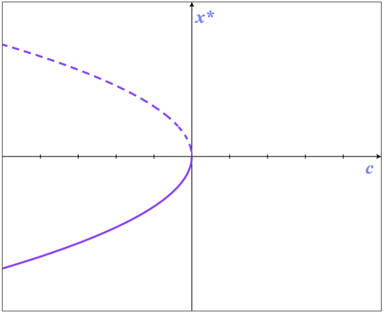
there are two ways to study the asymptotic stability: either by computing the sign of $f_c'(x^*)$ or studying the graph of f_c (see Biomathematics 1).



- if $c = 0$: x_0^* is a positive shunt,
- if $c < 0$: $x_-^* = -\sqrt{-c}$ is LAS and $x_+^* = \sqrt{-c}$ is unstable.

Saddle-node bifurcation

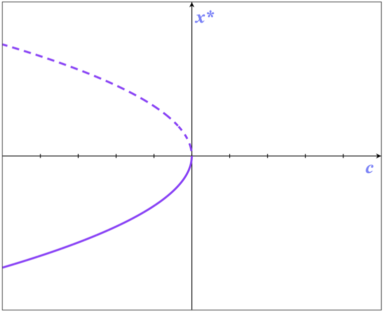
Bifurcation diagram:



- continuous line: $x^* = -\sqrt{-c}$ is the LAS branch,
- dashed line: $x^* = \sqrt{-c}$ is the unstable branch,
- there is a positive shunt when $c = 0$.

Saddle-node bifurcation

Bifurcation diagram:

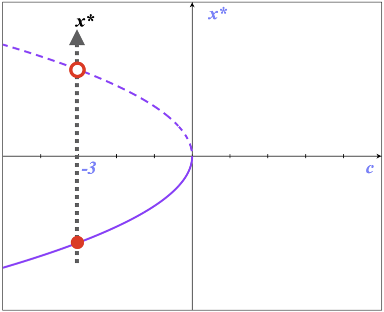


Saddle-node bifurcation at $c = 0$.

A **saddle-node bifurcation** occurs when two equilibria appear or disappear.

How do we read a bifurcation diagram ?

A vertical line gives us the phase portrait for a given value of c which allows us to deduce the chronicles (in exercise).



Detailed content

Transcritical bifurcation

Consider the following equation

$$x' = x^2 + cx \text{ and } c \in \mathbb{R},$$

with $f_c : x \mapsto x^2 + cx \in C^1(\mathbb{R})$.

Transcritical bifurcation

Equilibria:

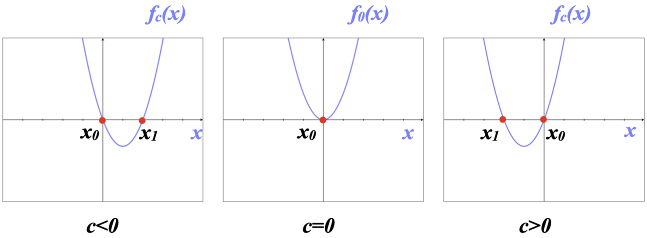
an equilibrium x^* satisfies $f_c(x^*) = 0$. That is

$$x^{*2} + cx = 0 \text{ which is equivalent to } x^* = 0 \text{ or } x^* = -c.$$

- Note that $x_0^* = 0$ is **always** an equilibrium regardless of the values of c.
- The second equilibrium exists also regardless the values of c, and we denote it $x_1^* = -c$.
- If $c = 0$, both equilibria overlap.

Transcritical bifurcation

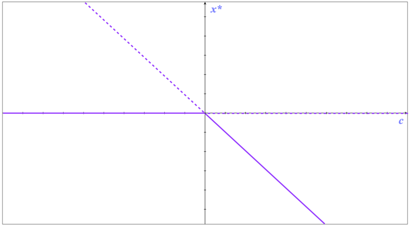
Stability:



- if $c < 0$: x_0^* is LAS, x_1^* is unstable,
- if $c = 0$: $x_0^* = x_1^*$ is a positive shunt,
- if $c > 0$: x_1^* is LAS, x_0^* is unstable.

Transcritical bifurcation

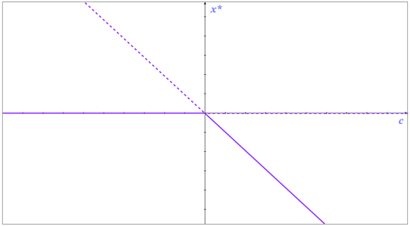
Bifurcation diagram:



- if $c < 0$ continuous line: $x^* = 0$ is the LAS branch, $x^* = -c$ is the unstable branch,
- if $c > 0$ continuous line: $x^* = -c$, is the LAS branch, $x^* = 0$ is the unstable branch.

Transcritical bifurcation

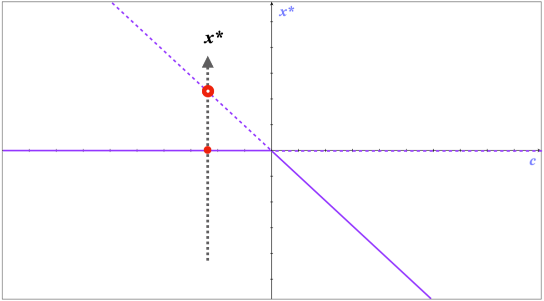
Bifurcation diagram:



Transcritical bifurcation at $c = 0$.
A **transcritical bifurcation** occurs when we keep the same number of equilibria but their stability changes.

How do we read a bifurcation diagram ?

A vertical line gives us the phase portrait for a given value c which allows us to deduce the chronicles.



Detailed content

Supercritical pitchfork bifurcation

Consider the following equation

$$x' = cx - x^3 \text{ and } c \in \mathbb{R},$$

with $f \in C^1(\mathbb{R})$ (for simplification we note $x = x(t)$).

Supercritical pitchfork bifurcation

Equilibria:

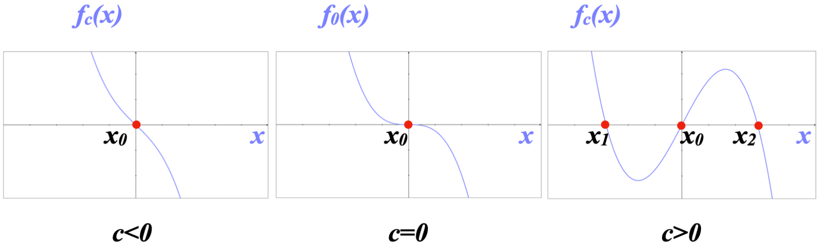
an equilibrium x^* satisfies $f_c(x^*) = 0$. That is

$$cx - x^{*3} = 0 \text{ which is equivalent to } x^* = 0 \text{ or } c - x^{*2} = 0.$$

- Note that $x_0^* = 0$ is **always** an equilibrium regardless of the values of c.
- The two other equilibria exist only if $c \geq 0$, and we denote them $x_1^* = -\sqrt{c}$ and $x_2^* = \sqrt{c}$.
- If $c = 0$, the three equilibria overlap.

Supercritical pitchfork bifurcation

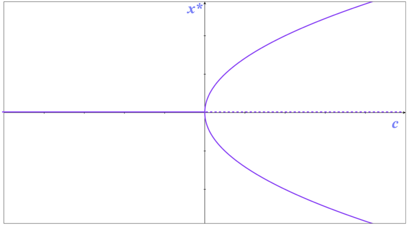
Stability:



- if $c < 0$: x_0^* is the only equilibrium and it is LAS,
- if $c = 0$: $x_0^* = x_1^* = x_2^*$ and they are LAS,
- if $c > 0$: x_0^* is unstable while x_1^* and x_2^* are LAS.

Supercritical pitchfork bifurcation

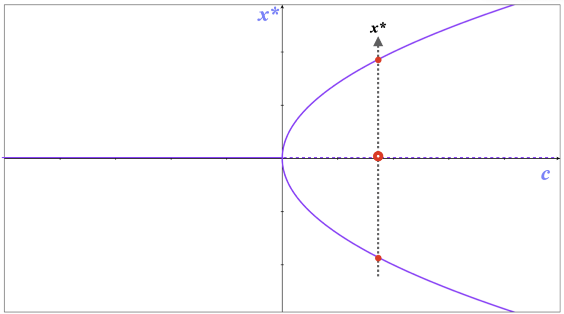
Bifurcation diagram:



Supercritical pitchfork bifurcation at $c = 0$.
A **supercritical pitchfork bifurcation** occurs when the bifurcation diagram looks (locally) like a pitchfork with the external branches being LAS.

How do we read a bifurcation diagram ?

A vertical line gives us the phase portrait for a given value c which allows us to deduce the chronicles.



Detailed content

Subcritical pitchfork bifurcation

Consider the following equation

$$x' = cx + x^3 \text{ and } c \in \mathbb{R},$$

with $f \in C^1(\mathbb{R})$ (for simplification we note $x = x(t)$).

Subcritical pitchfork bifurcation

Equilibria:

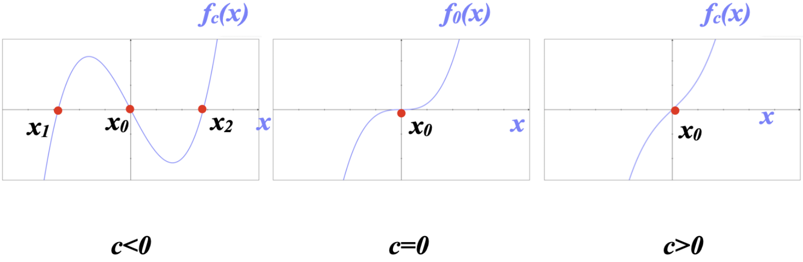
an equilibrium x^* satisfies $f_c(x^*) = 0$. That is

$$cx + x^{*3} = 0 \text{ which is equivalent to } x^* = 0 \text{ or } c + x^{*2} = 0.$$

- Note that $x_0^* = 0$ is **always** an equilibrium regardless of the values of c.
- The two other equilibria exist only if $c \leq 0$, and we denote them $x_1^* = -\sqrt{-c}$ and $x_2^* = \sqrt{-c}$.
- If $c = 0$, the three equilibria overlap.

Subcritical pitchfork bifurcation

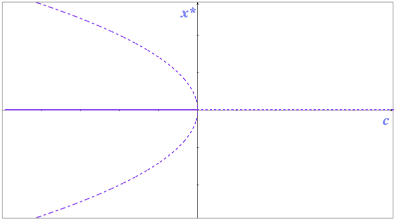
Stability:



- if $c > 0$: x_0^* is the only equilibrium and it is unstable,
- if $c = 0$: $x_0^* = x_1^* = x_2^*$ and they are unstable,
- if $c < 0$: x_0^* is LAS while x_1^* and x_2^* are unstable.

Subcritical pitchfork bifurcation

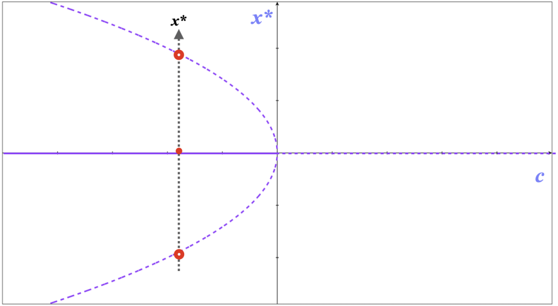
Bifurcation diagram:



Subcritical pitchfork bifurcation at $c = 0$.
A **subcritical pitchfork bifurcation** occurs when the bifurcation diagram looks (locally) like a pitchfork with the external branches being unstable.

How do we read a bifurcation diagram ?

A vertical line gives us the phase portrait for a given value c which allows us to deduce the chronicles.



Remarks:

- Supercritical pitchfork bifurcations are often called safe or soft because stability always exist regardless of the values of parameter c. And when crossing the bifurcation point, new solutions appear or disappear with low amplitudes.
- Subcritical pitchfork bifurcations are often called dangerous or hard because instability may be the only option for the equilibria. When crossing the bifurcation point, solutions may totally lose their stability with high amplitude. We call this a **catastrophe** in mathematics.
- In most biological problems, trajectories do not tend to infinity. There always exists a stabilizing effect.

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Hysteresis

Consider the following equation

$$x' = c + x - x^3 \text{ and } c \in \mathbb{R},$$

with $f \in C^1(\mathbb{R})$ (for simplification we note $x = x(t)$).

- Study the equilibria in function of parameter c,
- Study their stability in function of parameter c,
- Draw the bifurcation diagram and give an interpretation of the results.

Hysteresis

Hysteresis

Hysteresis

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Bifurcation with two parameters: the cusp case

Consider the following equation

$$x' = c + dx - x^3, \ c \text{ and } d \in \mathbb{R},$$

with $f \in C^1(\mathbb{R})$ (for simplification we note $x = x(t)$).

- Study the equilibria in function of parameters c and d,
- Study their stability in function of parameter c and d,
- Draw the bifurcation diagram and give an interpretation of the results.

The cusp case

The cusp case

The cusp case

The cusp case

The cusp case

The cusp case

The cusp case