English support for the course on difference equations

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For all details, refer to Elaydi S. 2005.

An introduction to difference equations

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Introduction

- Difference equations = recursion equations = discrete-time equations

- Used in modelling of biological phenomena
    - Population dynamics: non-overlapping generation like fish, plant or insect pop
    - Population genetics

- Use in numerically solving and simulating ODE or PDE
    - Euler scheme
    - Runge-Kutta

Numerical Solutions of Differential Equations

Euler's Method

Consider the first-order differential equation

$$x'(t) = g(t, x(t)), \qquad x(t_0) = x_0, \qquad t_0 \leq t \leq b. \tag{1.4.1}$$

Let us divide the interval $[t_0, b]$ into N equal subintervals. The size of each subinterval is called the *step size* of the method and is denoted by $h = (b - t_0)/N$. This step size defines the *nodes* $t_0, t_1, t_2, \ldots, t_N$, where $t_j = t_0 + jh$. Euler's method approximates $x'(t)$ by $(x(t+h) - x(t))/h$.

Substituting this value into (1.4.1) gives

$$x(t+h) = x(t) + hg(t, x(t)),$$

and for $t = t_0 + nh$, we obtain

$$x[t_0 + (n+1)h] = x(t_0 + nh) + hg[t_0 + nh, x(t_0 + nh)] \tag{1.4.2}$$

for $n = 0, 1, 2, \ldots, N-1$.

Adapting the difference equation notation and replacing $x(t_0 + nh)$ by $x(n)$ gives

$$x(n+1) = x(n) + hg[n, x(n)]. \tag{1.4.3}$$

Equation (1.4.3) defines *Euler's algorithm*, which approximates the solutions of the differential equation (1.4.1) at the node points.

Example 1.11. Let us now apply Euler's method to the differential equation:

$$x'(t) = 0.7x^2(t) + 0.7, \qquad x(0) = 1, \qquad t \in [0,1] \qquad (DE) \text{ (see footnote 3).}$$

Using the separation of variable method, we obtain

$$\frac{1}{0.7}\int \frac{dx}{x^2+1} = \int dt.$$

Hence

$$\tan^{-1}(x(t)) = 0.7t + c.$$

Letting $x(0) = 1$, we get $c = \frac{\pi}{4}$. Thus, the exact solution of this equation is given by $x(t) = \tan\left(0.7t + \frac{\pi}{4}\right)$.

The corresponding difference equation using Euler's method is

$$x(n+1) = x(n) + 0.7h(x^2(n) + 1), \qquad x(0) = 1 \qquad (\Delta E)$$

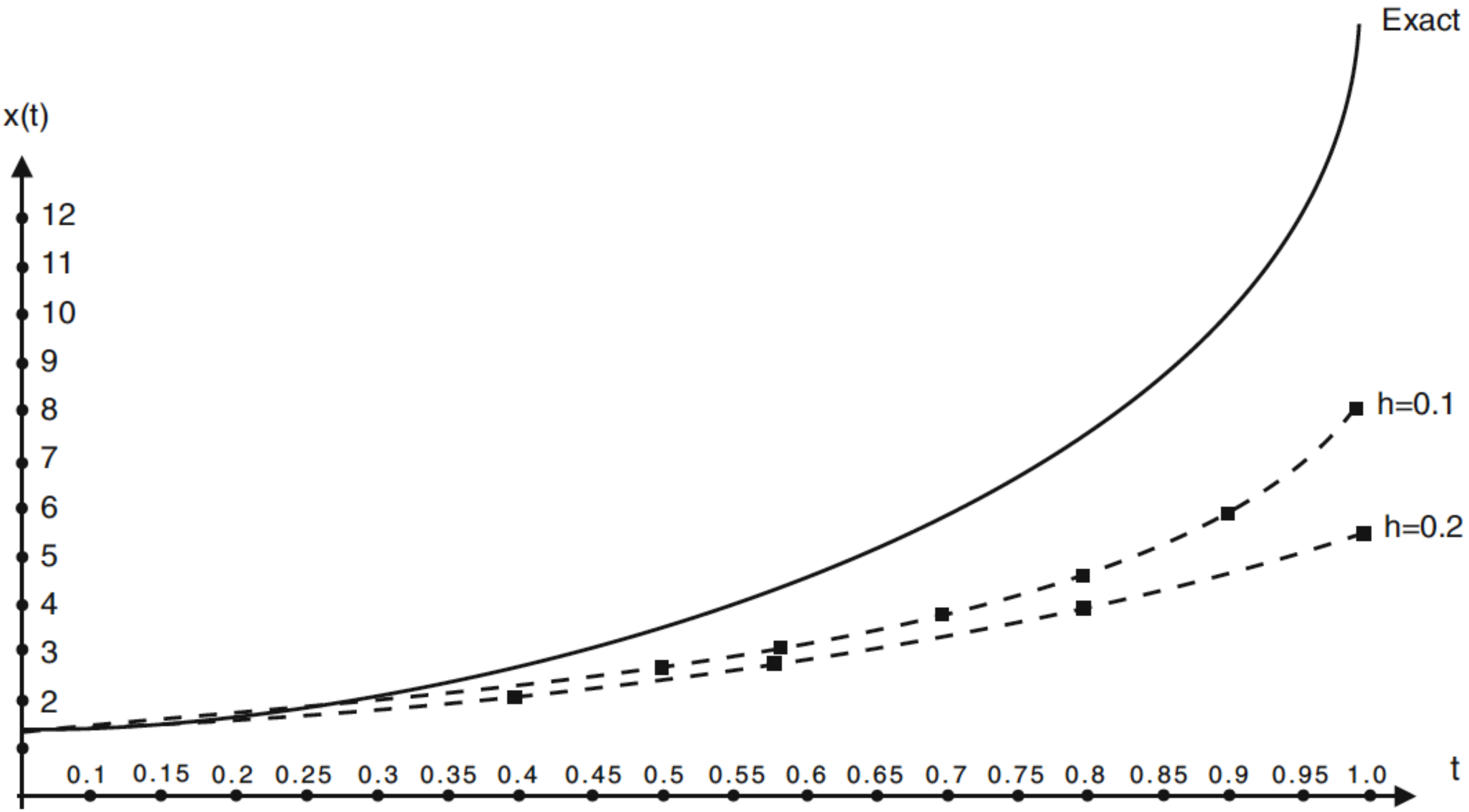


FIGURE 1.12. The $(n, x(n))$ diagram.

Definitions

Difference equations usually describe the evolution of certain phenomena over the course of time. For example, if a certain population has discrete generations, the size of the $(n+1)$st generation $x(n+1)$ is a function of the nth generation $x(n)$. This relation expresses itself in the *difference equation*

$$x(n+1) = f(x(n)). \tag{1.1.1}$$

We may look at this problem from another point of view. Starting from a point x_0, one may generate the sequence

$$x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \ldots .$$

For convenience we adopt the notation

$$f^2(x_0) = f(f(x_0)), \quad f^3(x_0) = f(f(f(x_0))), \quad \text{etc.}$$

$f(x_0)$ is called the *first iterate* of x_0 under f; $f^2(x_0)$ is called the second iterate of x_0 under f; more generally, $f^n(x_0)$ is the nth iterate of x_0 under f.

This iterative procedure is an example of a *discrete dynamical system*. Letting $x(n) = f^n(x_0)$, we have

$$x(n+1) = f^{n+1}(x_0) = f[f^n(x_0)] = f(x(n)),$$

Definitions (*continued*)

Definition A point x^* in the domain of f is said to be an *equilibrium point* of (1.1.1) if it is a fixed point of f, i.e., $f(x^*) = x^*$.

In other words, x^* is a *constant solution* of (1.1.1), since if $x(0) = x^*$ is an initial point, then $x(1) = f(x^*) = x^*$, and $x(2) = f(x(1)) = f(x^*) = x^*$, and so on.

Graphically, an equilibrium point is the x-coordinate of the point where the graph of f intersects the diagonal line $y = x$ (Figures 1.1). For example, there are three equilibrium points for the equation

$$x(n+1) = x^3(n)$$

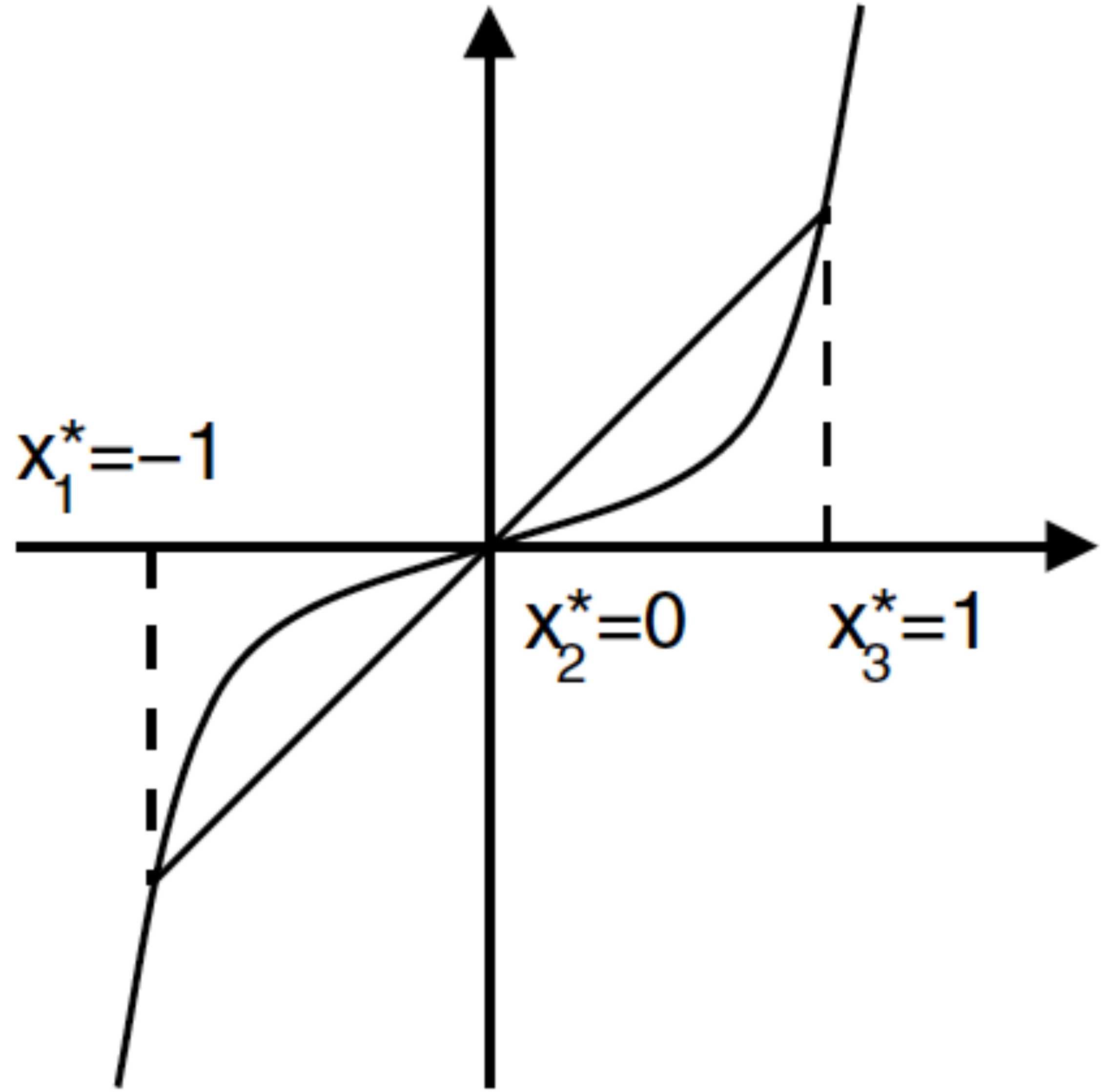


FIGURE 1.1. Fixed points of $f(x) = x^3$.

Definitions (*continued*)

Definition (a) The equilibrium point x^* of (1.1.1) is *stable* (Figure 1.4) if given $\varepsilon > 0$ there exists $\delta > 0$ such that $|x_0 - x^*| < \delta$ implies $|f^n(x_0) - x^*| < \varepsilon$ for all $n > 0$. If x^* is not stable, then it is called *unstable* (Figure 1.5).

(b) The point x^* is said to be *attracting* if there exists $\eta > 0$ such that

$$|x(0) - x^*| < \eta \quad \text{implies} \quad \lim_{n\to\infty} x(n) = x^*.$$

If $\eta = \infty$, x^* is called a *global attractor* or *globally attracting.*

(c) The point x^* is an *asymptotically stable equilibrium point* if it is stable and attracting (Figure 1.6).

If $\eta = \infty$, x^* is said to be *globally asymptotically stable* (Figure 1.7).

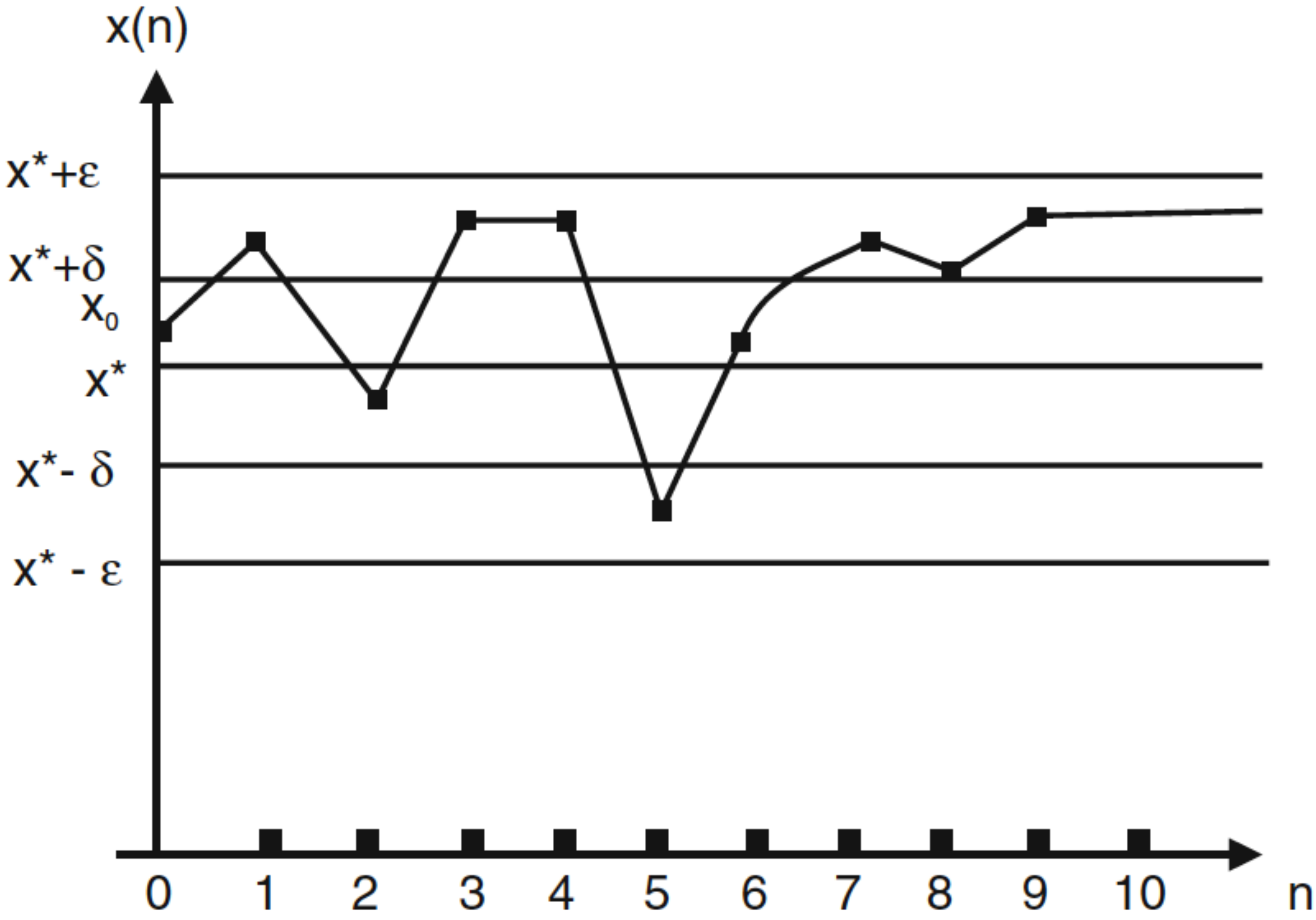


FIGURE 1.4. Stable x^*. If $x(0)$ is within δ from x^*, then $x(n)$ is within ε from $x(n)$ for all $n > 0$.

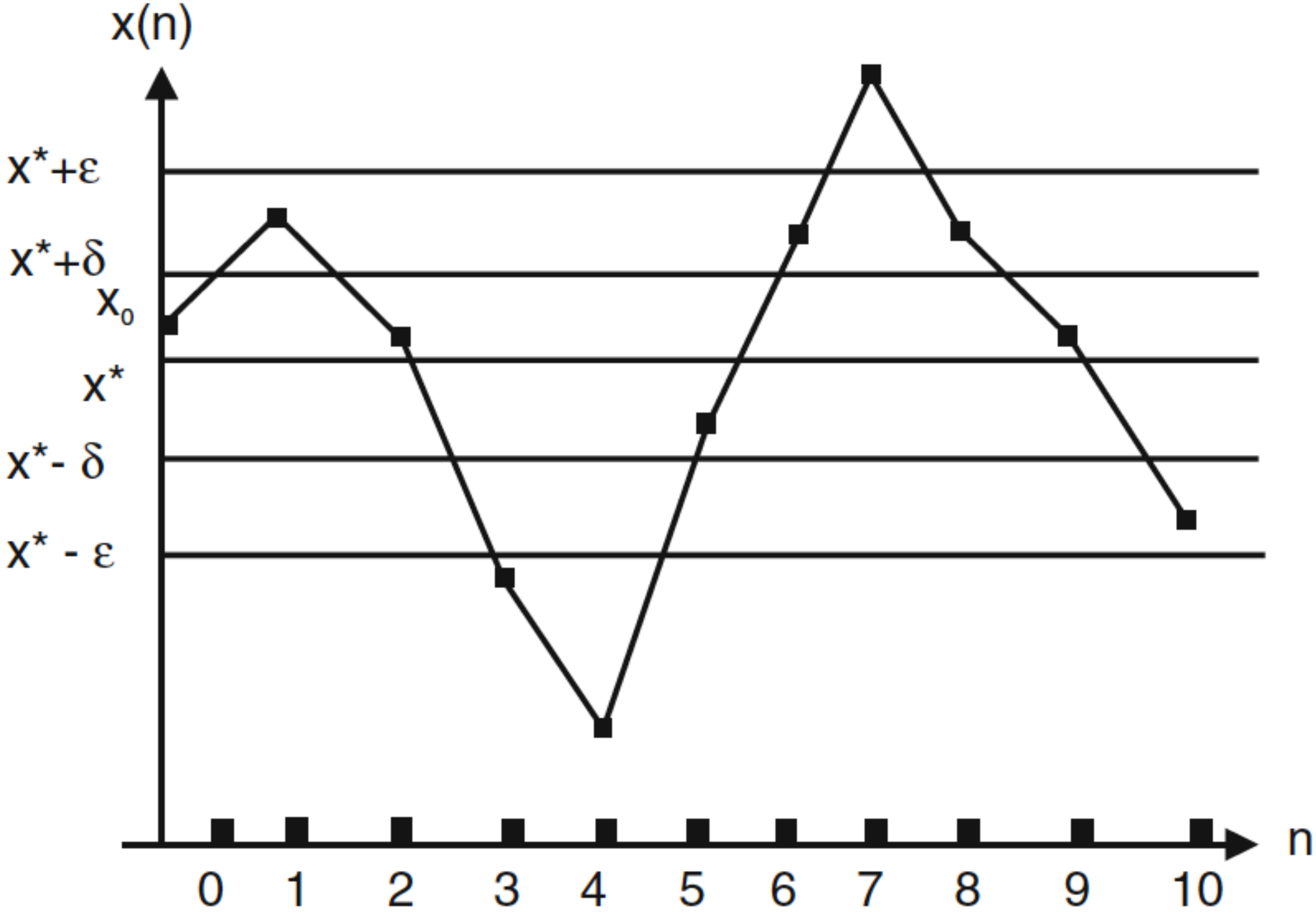


FIGURE 1.5. Unstable x^*. There exists $\varepsilon > 0$ such that no matter how close $x(0)$ is to x^*, there will be an N such that $x(N)$ is at least ε from x^*.

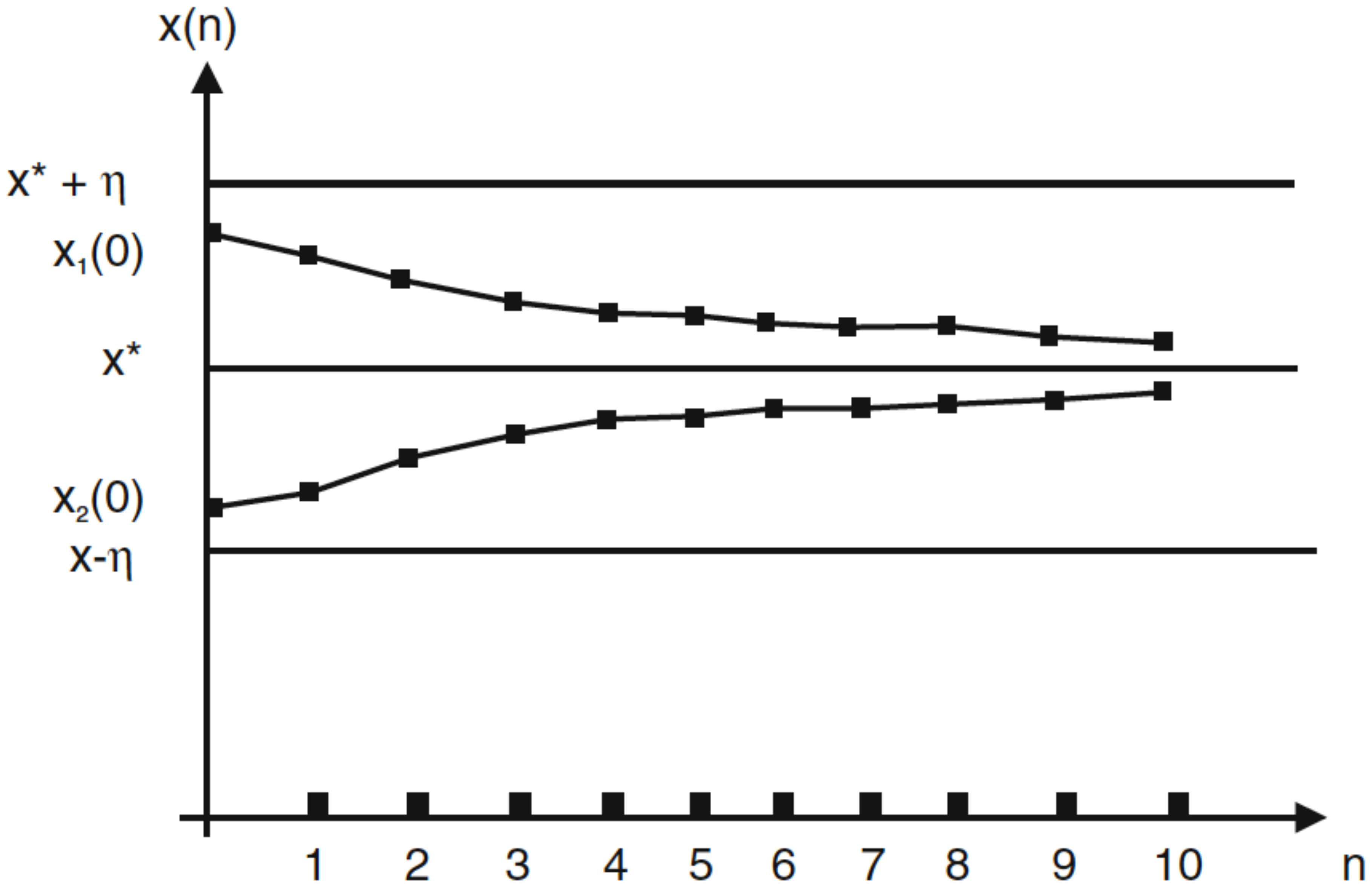


FIGURE 1.6. Asymptotically stable x^*. Stable if $x(0)$ is within η of x^*; then $\lim_{n\to\infty} x(n) = x^*$.

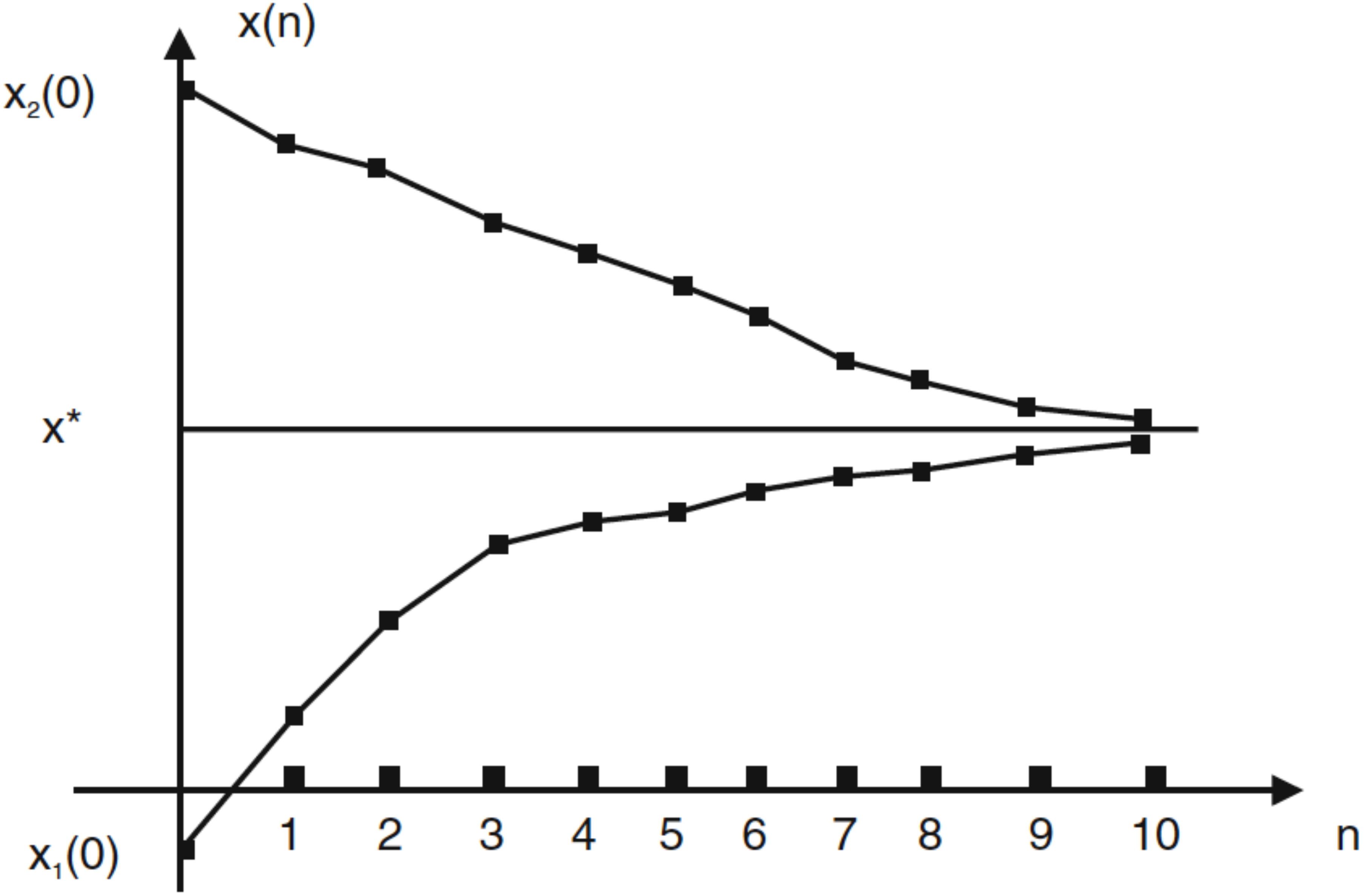


FIGURE 1.7. Globally asymptotically stable x^*. Stable and $\lim_{n\to\infty} x(n) = x^*$ for all $x(0)$.

The Stair Step (Cobweb) Diagram

We now give, in excruciating detail, another important graphical method for analyzing the stability of equilibrium (and periodic) points for (1.1.1). Since $x(n+1) = f(x(n))$, we may draw a graph of f in the $(x(n), x(n+1))$ plane. Then, given $x(0) = x_0$, we pinpoint the value $x(1)$ by drawing a vertical line through x_0 so that it also intersects the graph of f at $(x_0, x(1))$. Next, draw a horizontal line from $(x_0, x(1))$ to meet the diagonal line $y = x$ at the point $(x(1), x(1))$. A vertical line drawn from the point $(x(1), x(1))$ will meet the graph of f at the point $(x(1), x(2))$. Continuing this process, one may find $x(n)$ for all $n > 0$.

Example: the logistic equation

$$x(n+1) = \mu x(n)(1 - x(n)) = f(x(n)). \tag{1.3.4}$$

This equation is the simplest nonlinear first-order difference equation, commonly referred to as the (discrete) logistic equation.

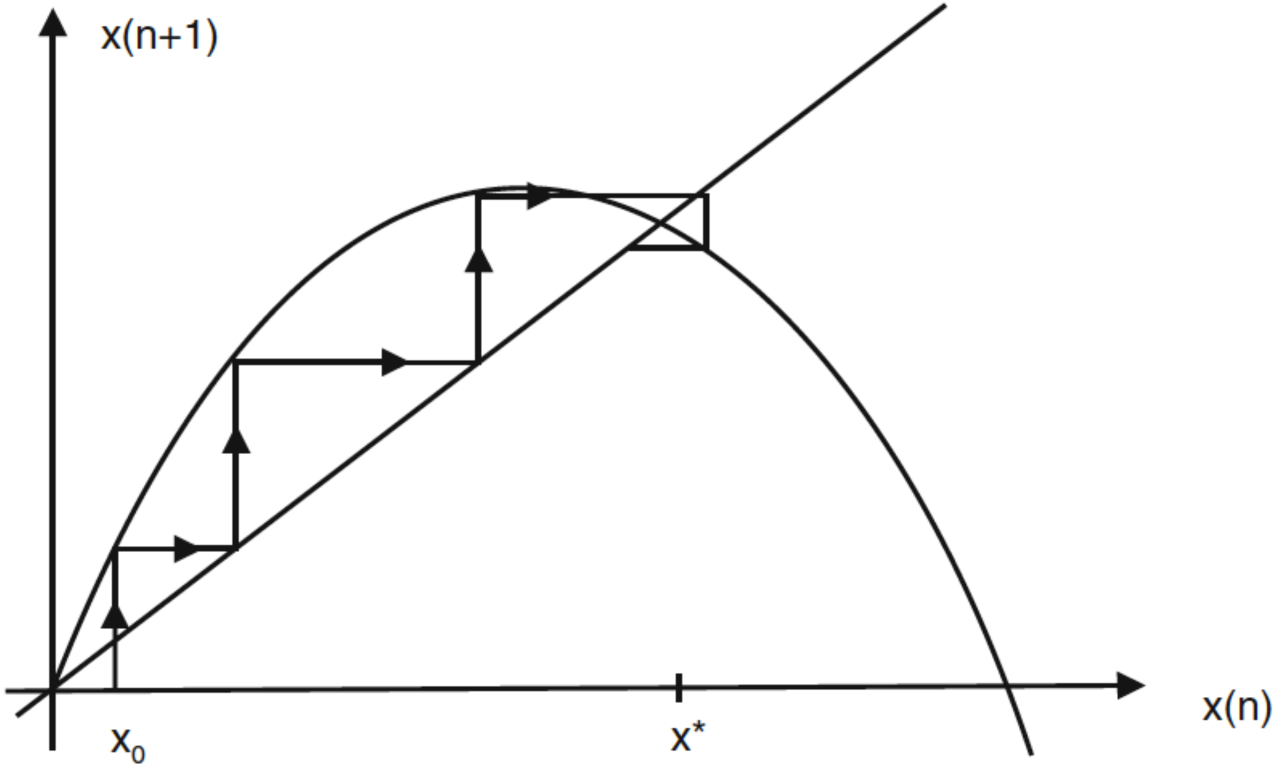


FIGURE 1.8. Stair step diagram for $\mu = 2.5$.

Linear difference equations

$$x_{n+1} = \lambda x_n \qquad \lambda \in \mathbb{R}$$

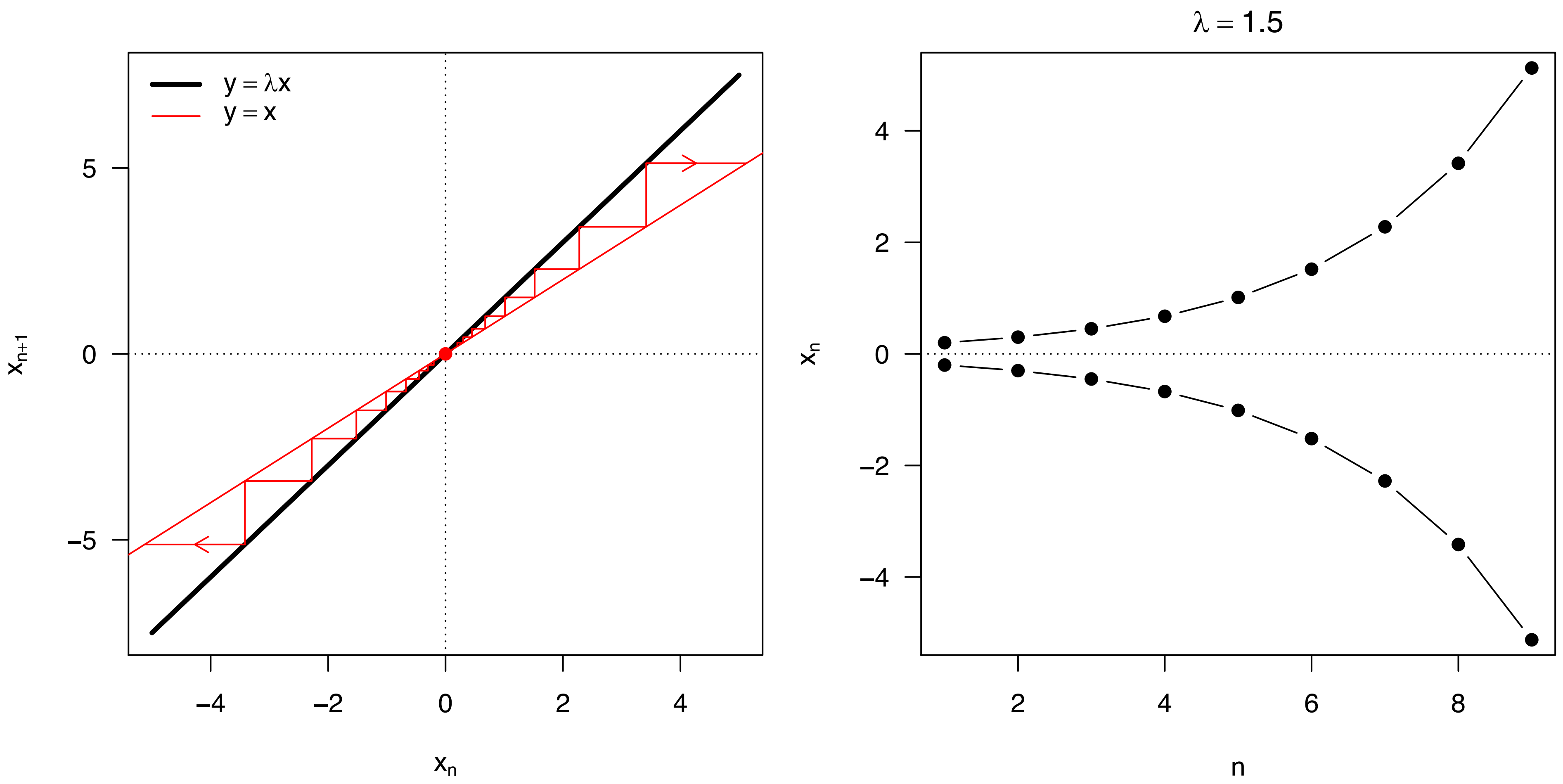
Fixed point

$$x^* = \lambda x^* \Leftrightarrow (1 - \lambda)\, x^* = 0 \Leftrightarrow x^* = 0 \quad si \quad \lambda \neq 1$$

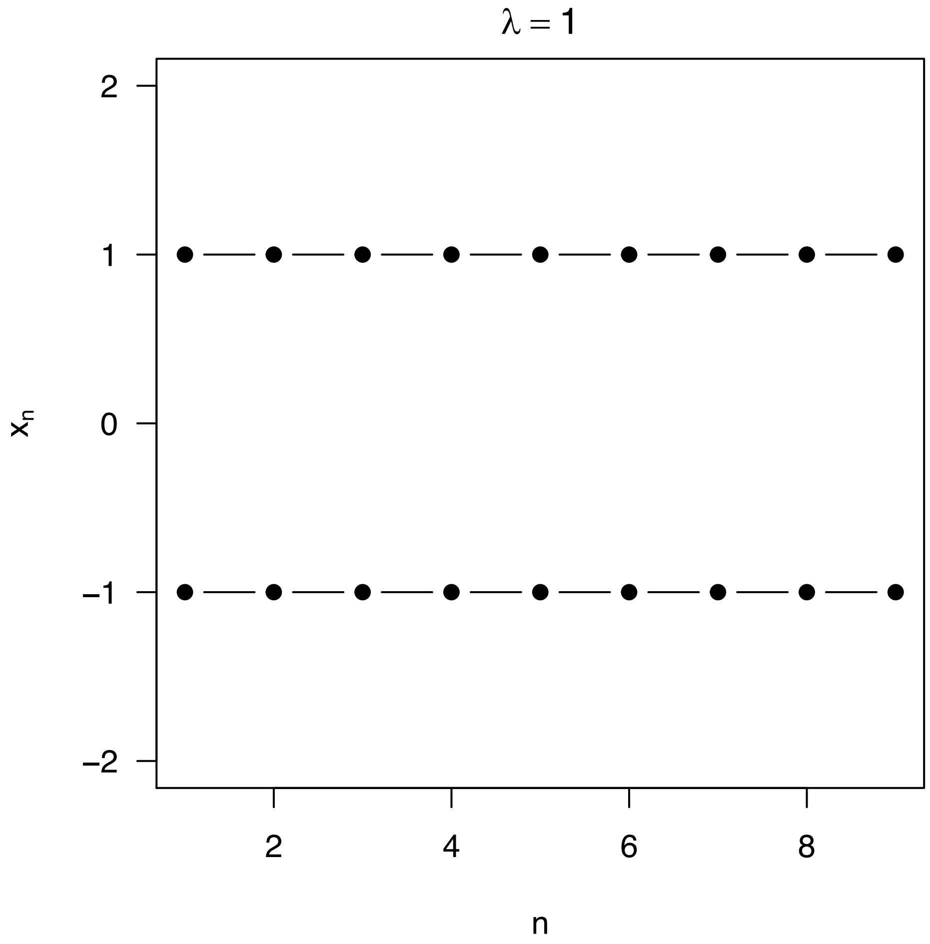
Solution

$$x_n = \lambda^n x_0$$

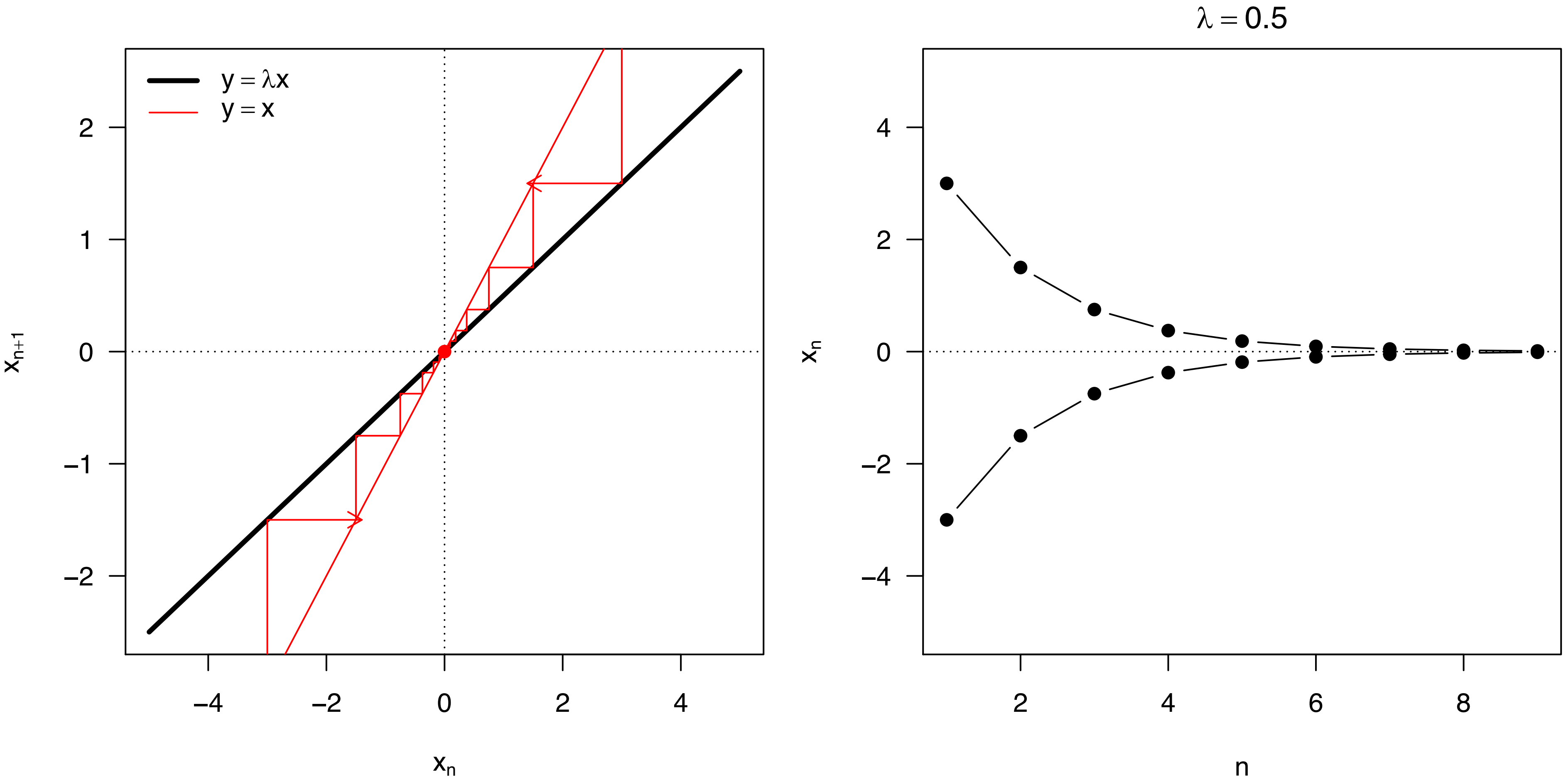
$\lambda > 1$



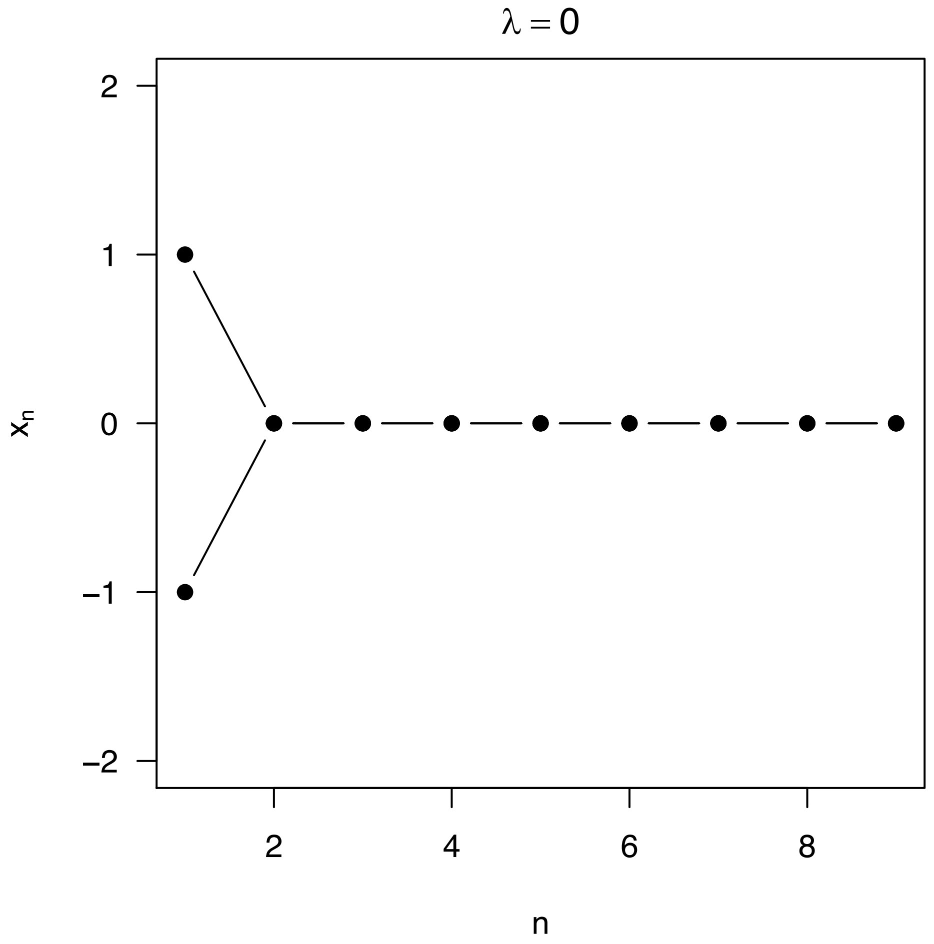
$\lambda = 1$



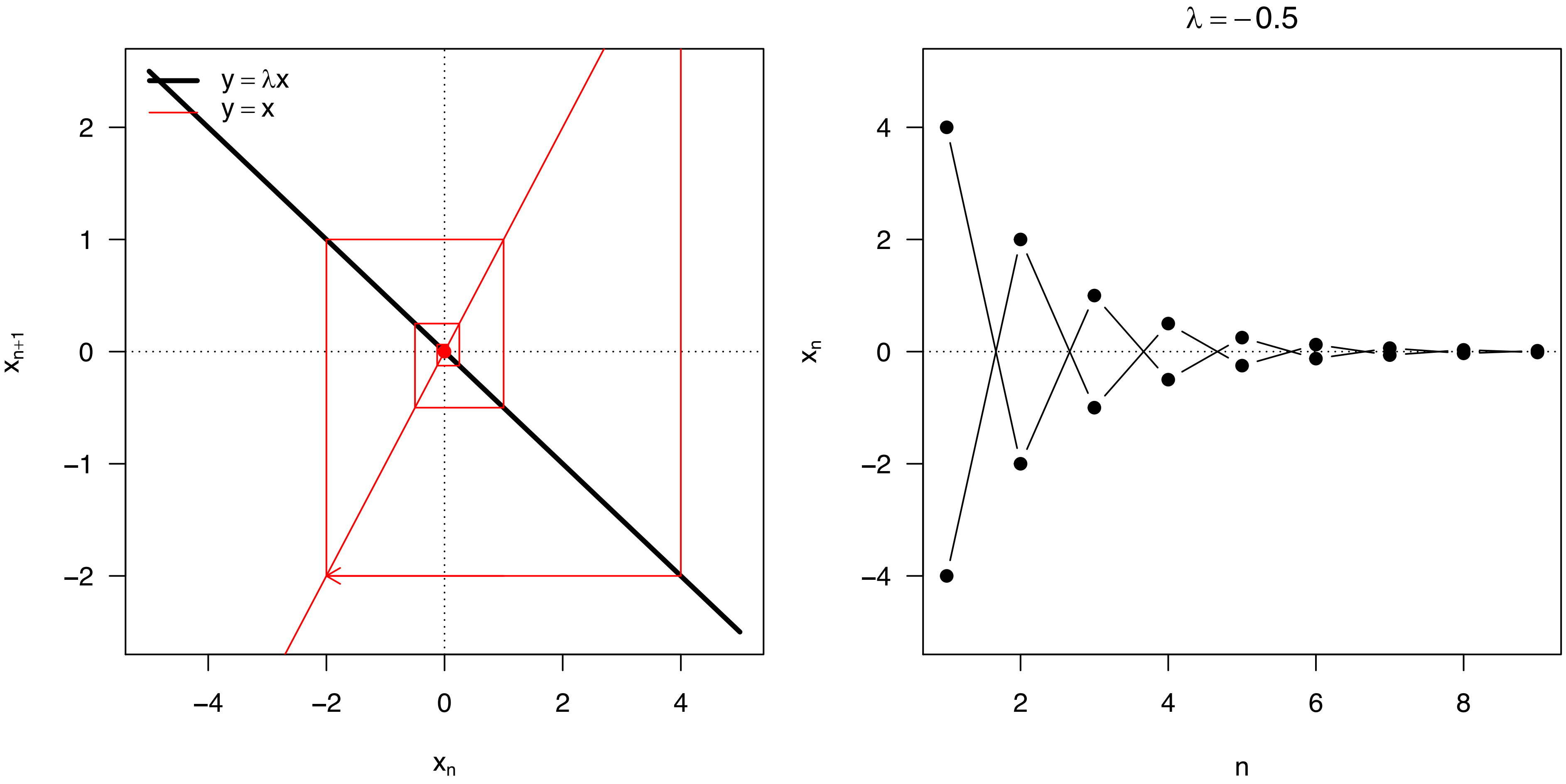
$0 < \lambda < 1$



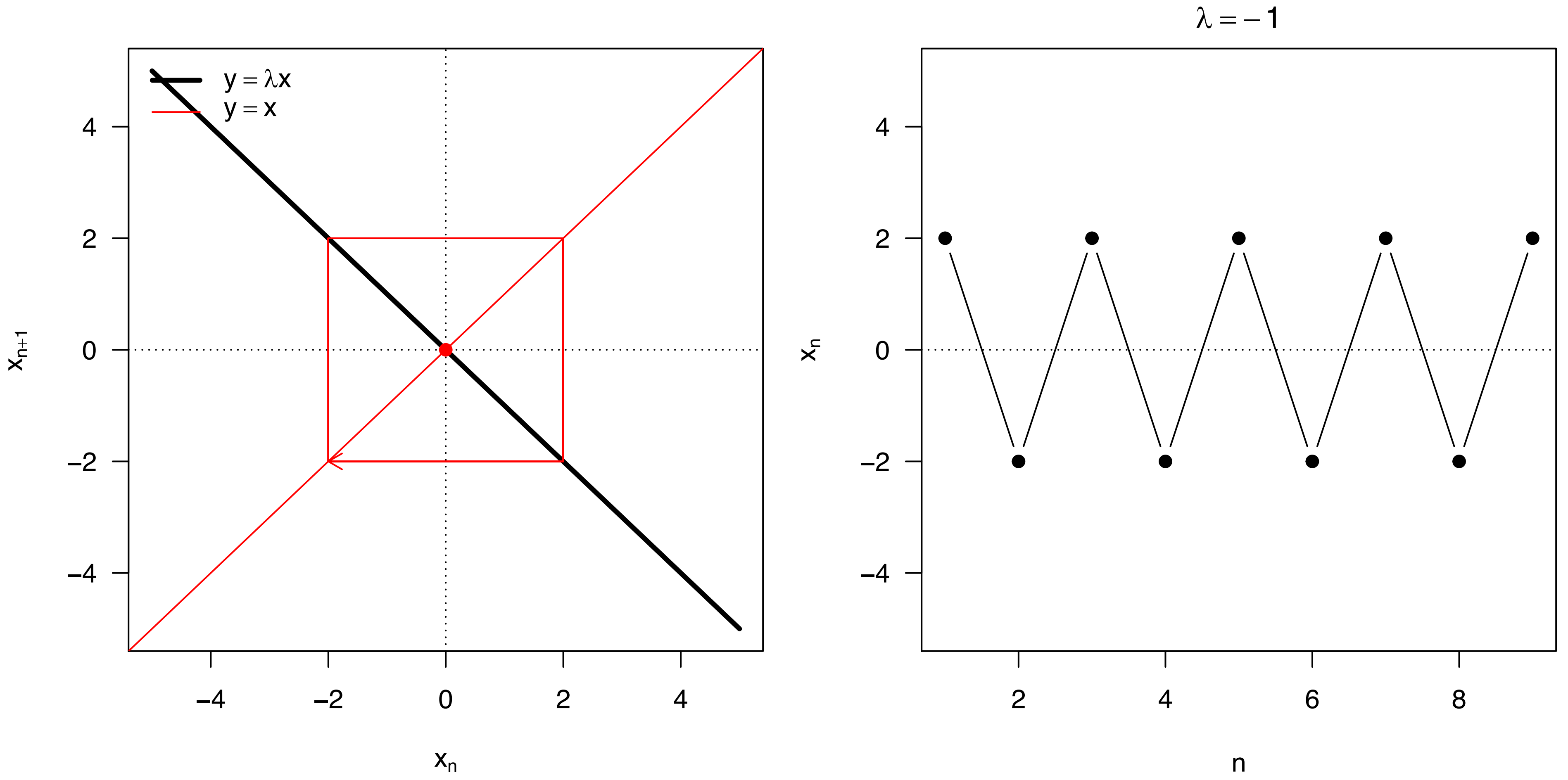
$\lambda = 0$



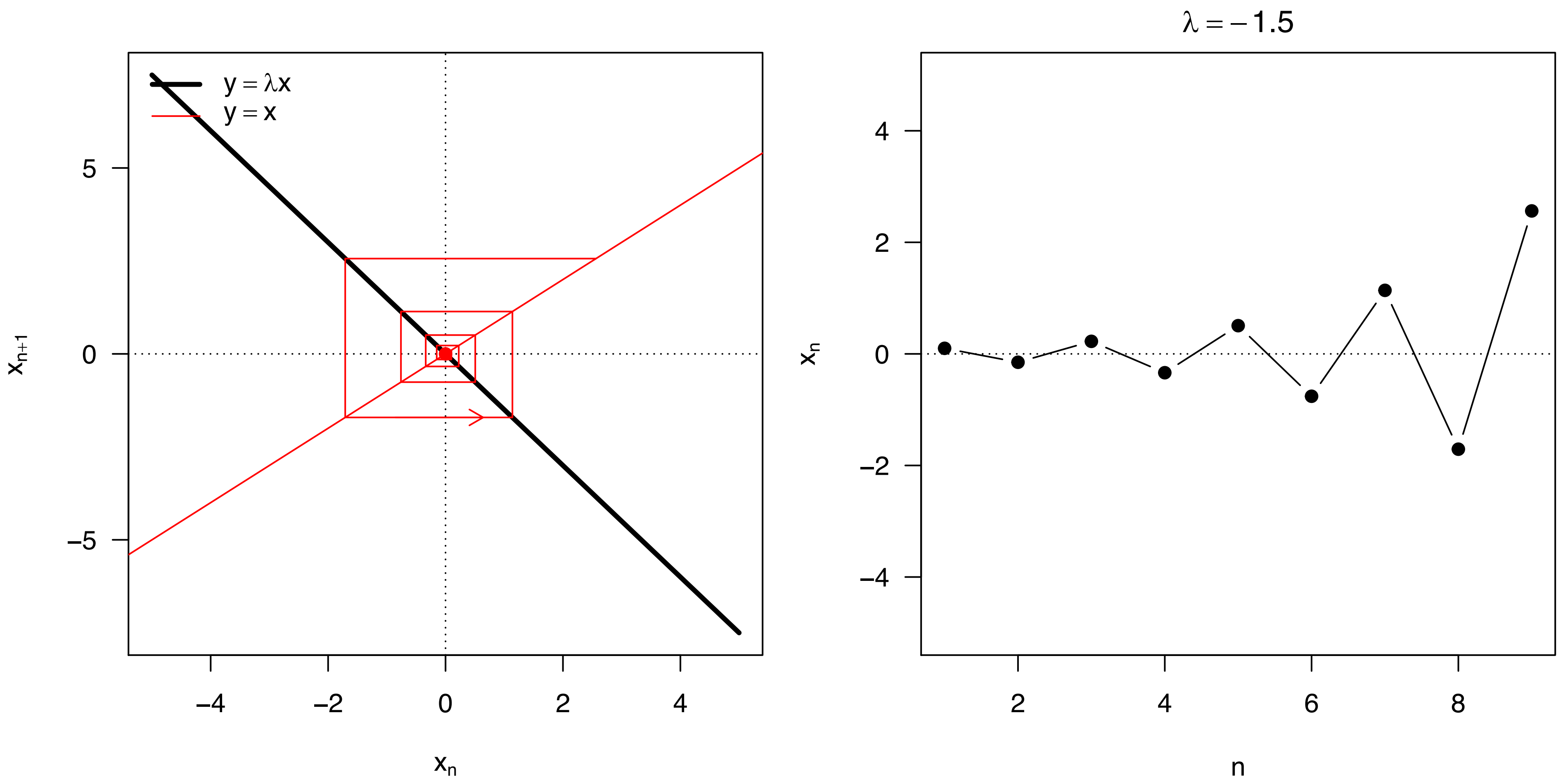
$-1 < \lambda < 0$



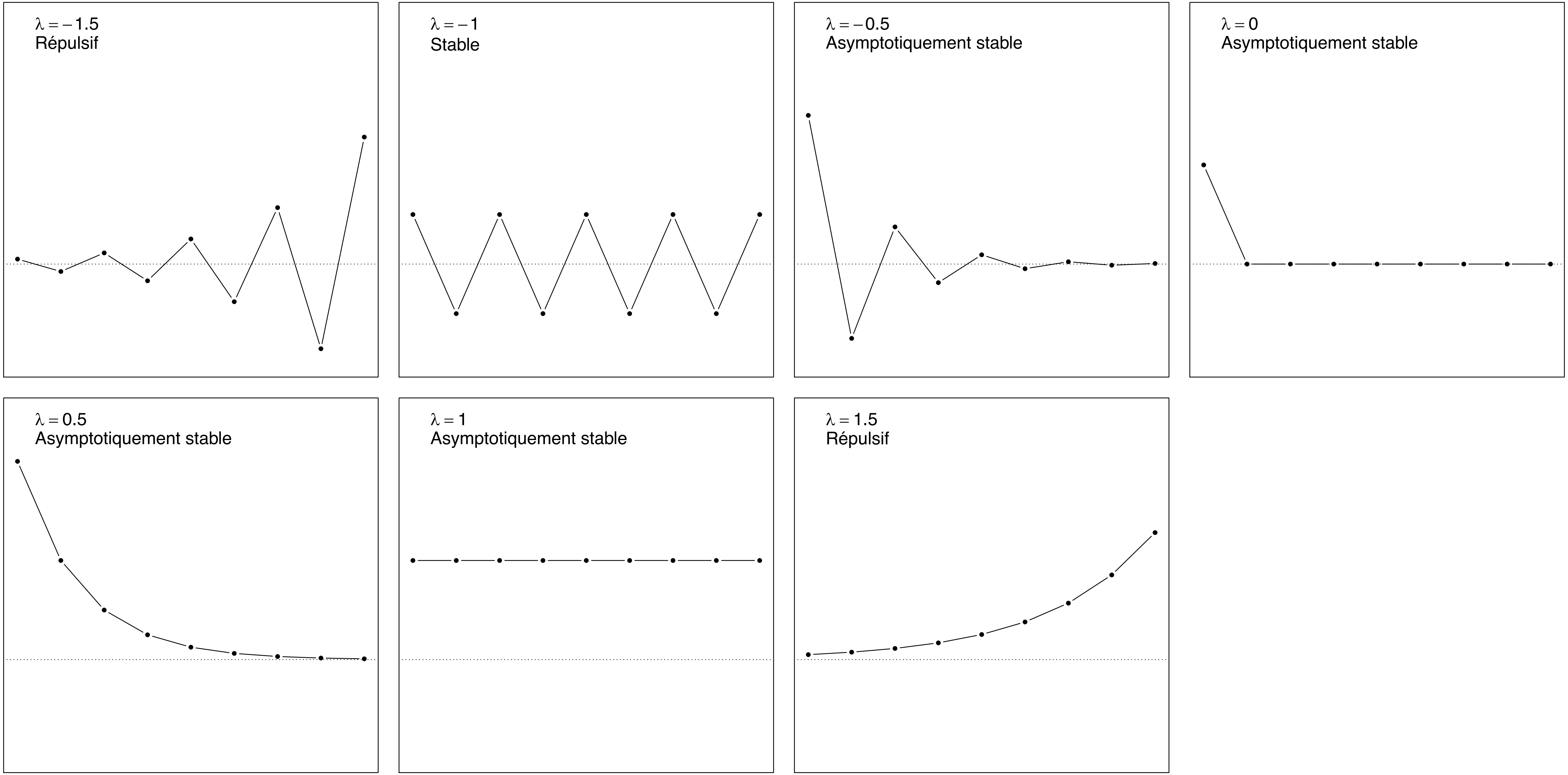
λ = - 1



λ < - 1



Summary


λ = −1.5
Répulsif
λ = −1
Stable
λ = −0.5
Asymptotiquement stable
λ = 0
Asymptotiquement stable
λ = 0.5
Asymptotiquement stable
λ = 1
Asymptotiquement stable
λ = 1.5
Répulsif

Non linear difference equations

$$x_{n+1} = f(x_n) \qquad\qquad x^* = f(x^*)$$

Linearization $\quad x_n \in V(x^*)$

$$u_n = x_n - x^*$$

$$f(x_n) \simeq f(x^*) + \left.\frac{df}{dx_n}\right|_{x_n = x^*} (x_n - x^*)$$

$$u_{n+1} + x^* \simeq f(x^*) + \left.\frac{df}{dx_n}\right|_{*} u_n$$

$$u_{n+1} \simeq \left.\frac{df}{dx_n}\right|_{*} u_n = \lambda^* u_n$$

Theorem *Let x^* be an equilibrium point of the difference equation*

$$x(n+1) = f(x(n)),$$

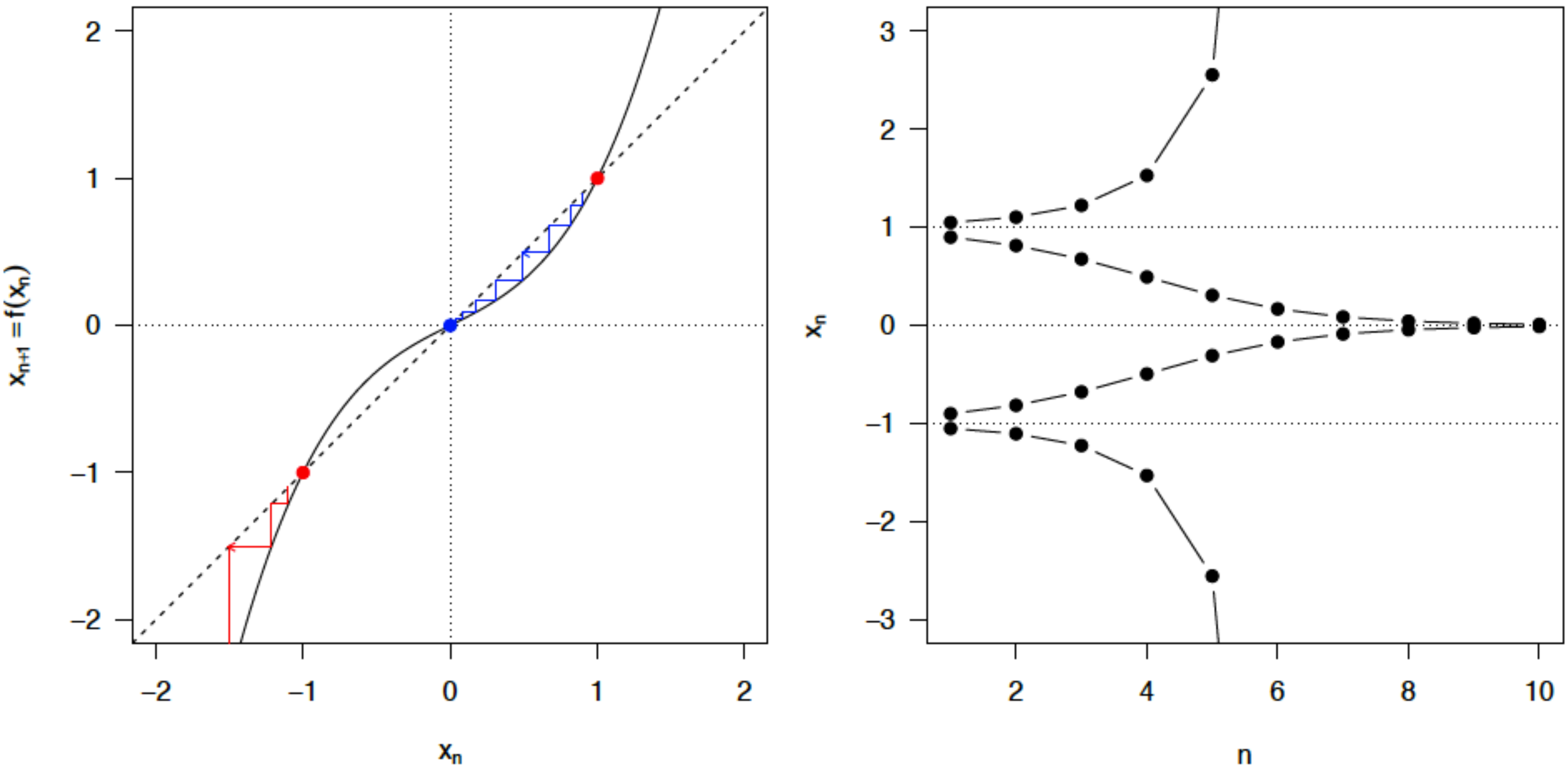
where f is continuously differentiable at x^*. The following statements then hold true:

(i) *If $|f'(x^*)| < 1$, then x^* is asymptotically stable.*

(ii) *If $|f'(x^*)| > 1$, then x^* is unstable.*

If $|f'(x^*)| < 1$, then x^* is said ***hyperbolic***.

$$x_{n+1} = \frac{1}{2}\left(x_n^3 + x_n\right)$$



Theorem *Suppose that for an equilibrium point* x^* $f'(x^*) = 1$. *The following statements then hold:*

(i) *If* $f''(x^*) \neq 0$, *then* x^* *is unstable.*

(ii) *If* $f''(x^*) = 0$ *and* $f'''(x^*) > 0$, *then* x^* *is unstable.*

(iii) *If* $f''(x^*) = 0$ *and* $f'''(x^*) < 0$, *then* x^* *is asymptotically stable.*

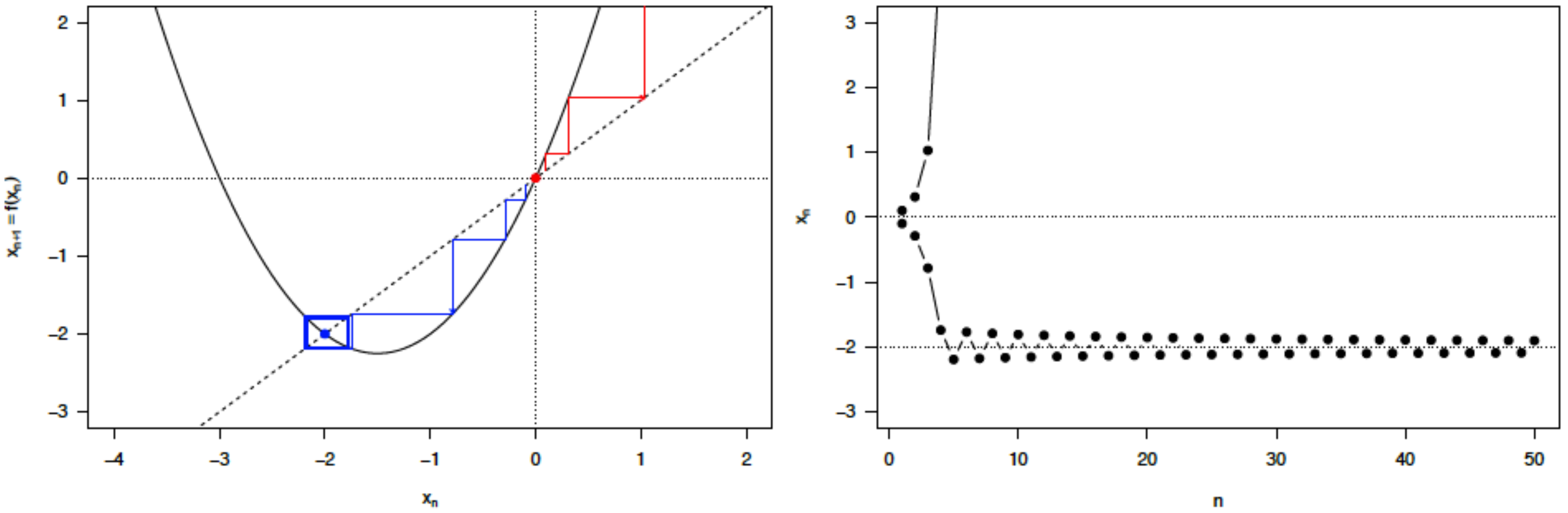
$$\boxed{Sf(x^*) = -f'''(x^*) - \frac{3}{2}\left(f''(x^*)\right)^2}$$

Theorem *Suppose that for the equilibrium point* x^* $f'(x^*) = -1$. *The following statements then hold:*

(i) *If* $Sf(x^*) < 0$, *then* x^* *is asymptotically stable.*

(ii) *If* $Sf(x^*) > 0$, *then* x^* *is unstable.*

$$x_{n+1} = x_n^2 + 3x_n$$



Discrete demographic models

- Populations are genetically compatible individuals of a same species (or a same sub-species, in a given place and which reproduce between them.

- Individuals may be uni- (such as microorganisms) or multi-cellular.

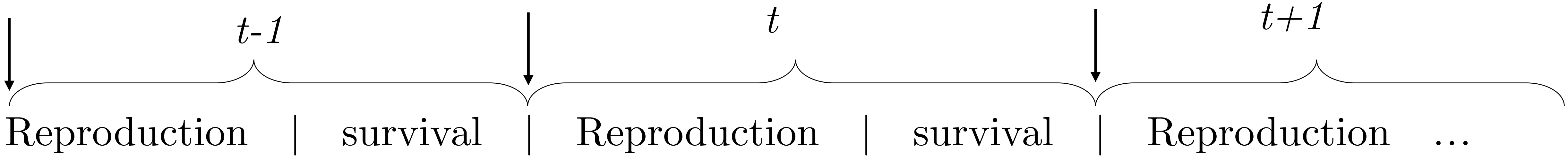
- Growth is ensured by:
  - Reproduction (sexual or not)
  - Individual survival

- Let be n_t the discrete nbr of individuals at time t.

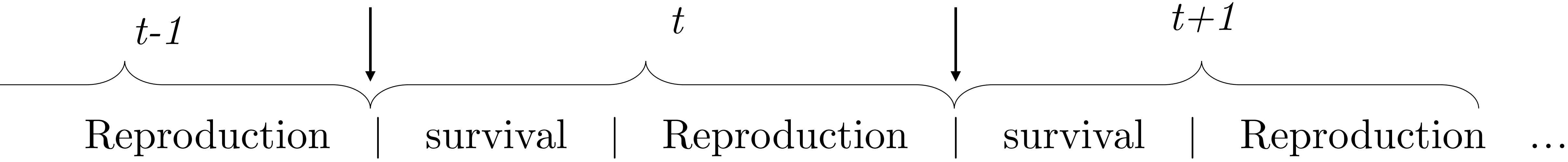
- n_t evolves with time according to births and deaths.

The exponential model (linear)

- A key issue with discrete-time models is the choice of the beginning of the time step.
- Two classical ways of doing:
  - Pre-breeding census



  - Post-breeding census



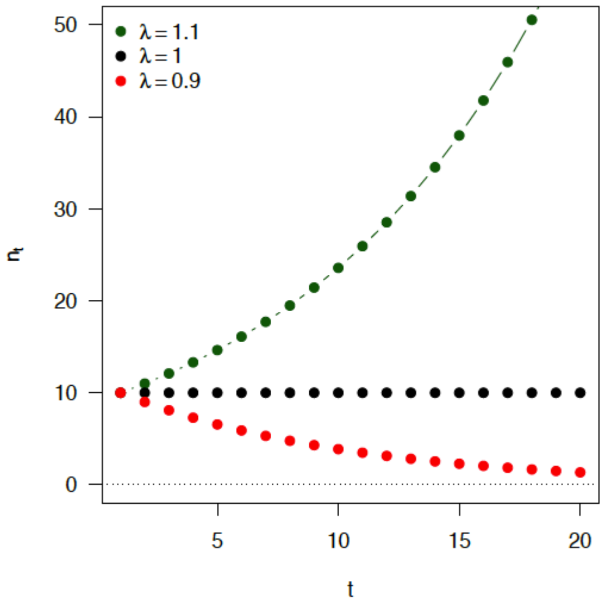
Example with a semelparus species

- Semelparus = one reproduction event per life span (e.g. annual plants)
- Hyp: pre-breeding census
- p_t: nbr of offspring at time t
- n_t: nbr of females at time t

$$p_{t+1} = f n_t \qquad n_{t+1} = s\left(1-m\right)p_{t+1} \qquad n_{t+1} = s\left(1-m\right)f n_t$$

$$\lambda = sf\left(1-m\right) > 0$$

$$n_t = \lambda^t n_0$$



$$\lambda = e^{r\delta} \Leftrightarrow r = \frac{\ln \lambda}{\delta}$$

Example with microbial population

- In the laboratory, under favourable conditions, a growing bacterial population doubles at regular intervals.

- Growth is by geometric progression: 1, 2, 4, 8, etc. or 2^0, 2^1, 2^2, 2^3.........2^n (where n = the number of generations)

- $x_n = 2^n \ x_0$

- $\lambda = 2 > 1$: bacteria are growing fast.

Generalization

$$n_{t+1} - n_t = an_t - bn_t \qquad n_{t+1} = \lambda n_t \qquad \lambda = 1 + a - b$$

- Bacterial growth

  Cellular death is neglected, 1 cell gives 2 new cells: $a = 1$ and $b = 0$, thus $\lambda = 2$

- Semelparus species

  $a = s\ (1 - m)\ f$ and $b = 1$, thus $\lambda = s\ (1 - m)\ f$

- Iteroparus species

  $b < 1$

The discrete-time logistic model

May RM. 1976. Simple mathematical models with very complicated dynamics. *Nature*:459–467. doi:10.1038/261459a0.

$$a(n_t) = \alpha_0 - \alpha n_t$$

$$b(n_t) = \beta_0 + \beta n_t$$

$$n_{t+1} - n_t$$

$$= a(n_t)\, n_t - b(n_t)\, n_t$$

$$= (\alpha_0 - \alpha n_t - \beta_0 - \beta n_t)\, n_t$$

$$= (\rho - (\alpha + \beta)\, n_t)\, n_t$$

$$= \rho n_t \left(1 - \frac{\alpha+\beta}{\rho} n_t\right)$$

$$K = \frac{\rho}{\alpha+\beta} \qquad n_{t+1} - n_t = \rho n_t \left(1 - \frac{n_t}{K}\right)$$

$$n_{t+1} = n_t + \rho n_t \left(1 - \frac{n_t}{K}\right)$$

$$n_{t+1} = n_t \left(1 + \rho - \frac{\rho n_t}{K}\right)$$

$$n_{t+1} = n_t \left(\lambda - \frac{\rho n_t}{K}\right)$$

$$n_{t+1} = \lambda n_t \left(1 - \frac{\rho n_t}{\lambda K}\right)$$

$$x_t = \frac{\rho n_t}{\lambda K}$$

$$x_{t+1} = \lambda x_t (1 - x_t)$$

$\lambda > 1$

The logistic equation and Bifurcation

$$x(n+1) = \mu x(n)[1 - x(n)], \tag{1.7.1}$$

which arises from iterating the function

$$F_\mu(x) = \mu x(1-x), \qquad x \in [0,1], \qquad \mu > 0. \tag{1.7.2}$$

To find the equilibrium points (fixed points of F_μ) of (1.7.1) we solve the equation

$$F_\mu(x^*) = x^*.$$

Hence the fixed points are $0, x^* = (\mu - 1)/\mu$. Next we investigate the stability of each equilibrium point separately.

(a) The equilibrium point 0. (See Figure 1.32.) Since $F'_\mu(0) = \mu$, it follows

(i) 0 is an asymptotically stable fixed point for $0 < \mu < 1$,

(ii) 0 is an unstable fixed point for $\mu > 1$.

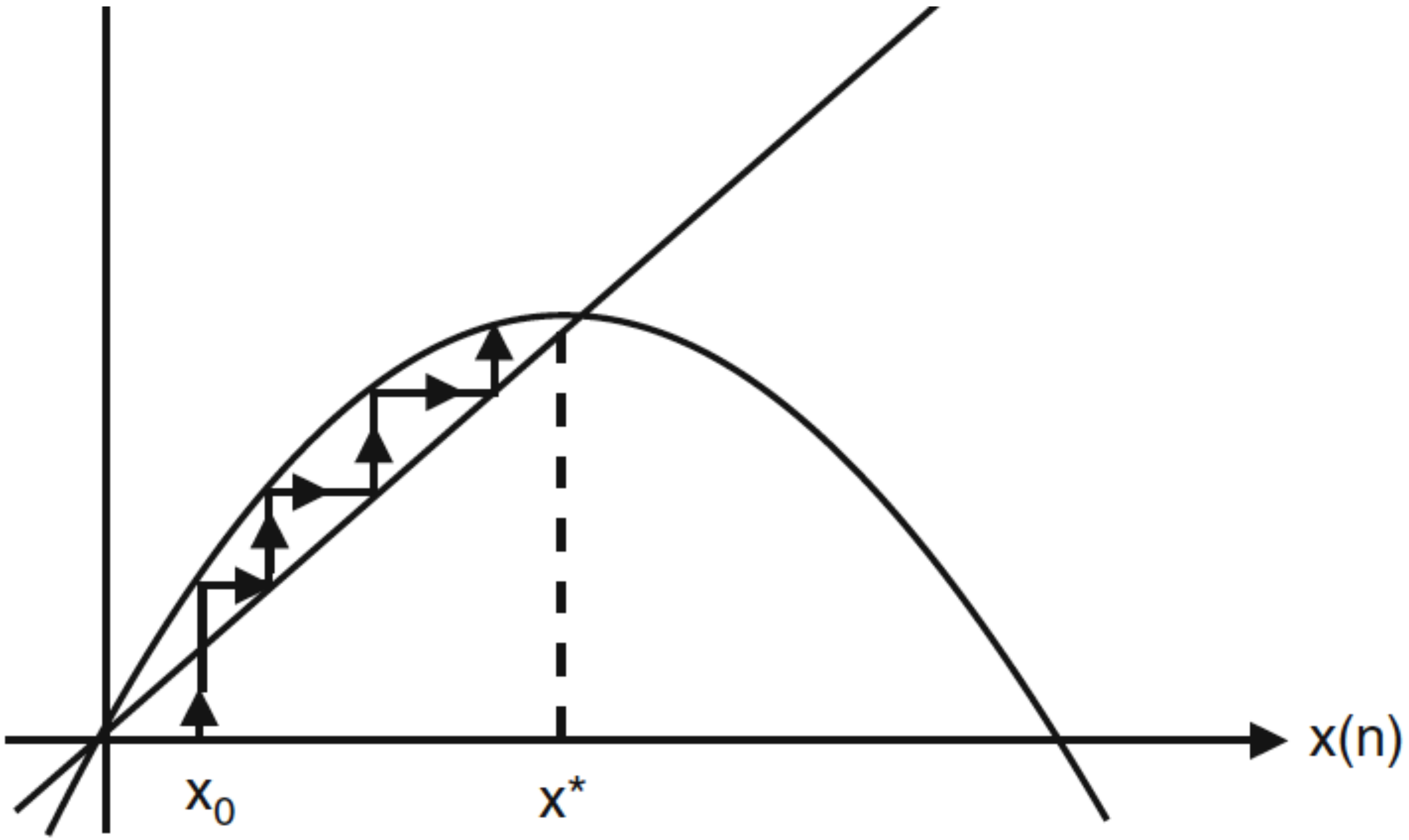


FIGURE 1.32. $\mu > 1$: 0 is an unstable fixed point, x^* is an asymptotically fixed point.

The logistic equation and Bifurcation (*continued*)

(b) The equilibrium point $x^* = (\mu - 1)/\mu, \mu \neq 1$. (See Figures 1.32, 1.33.)

In order to have $x^* \in (0, 1]$ we require that $\mu > 1$. Now, $F'_\mu((\mu - 1)/\mu) = 2 - \mu$. we obtain the following conclusions:

(i) x^* is an asymptotically stable fixed point for $1 < \mu \leq 3$ (Figure 1.32).

(ii) x^* is an unstable fixed point for $\mu > 3$ (Figure 1.33).

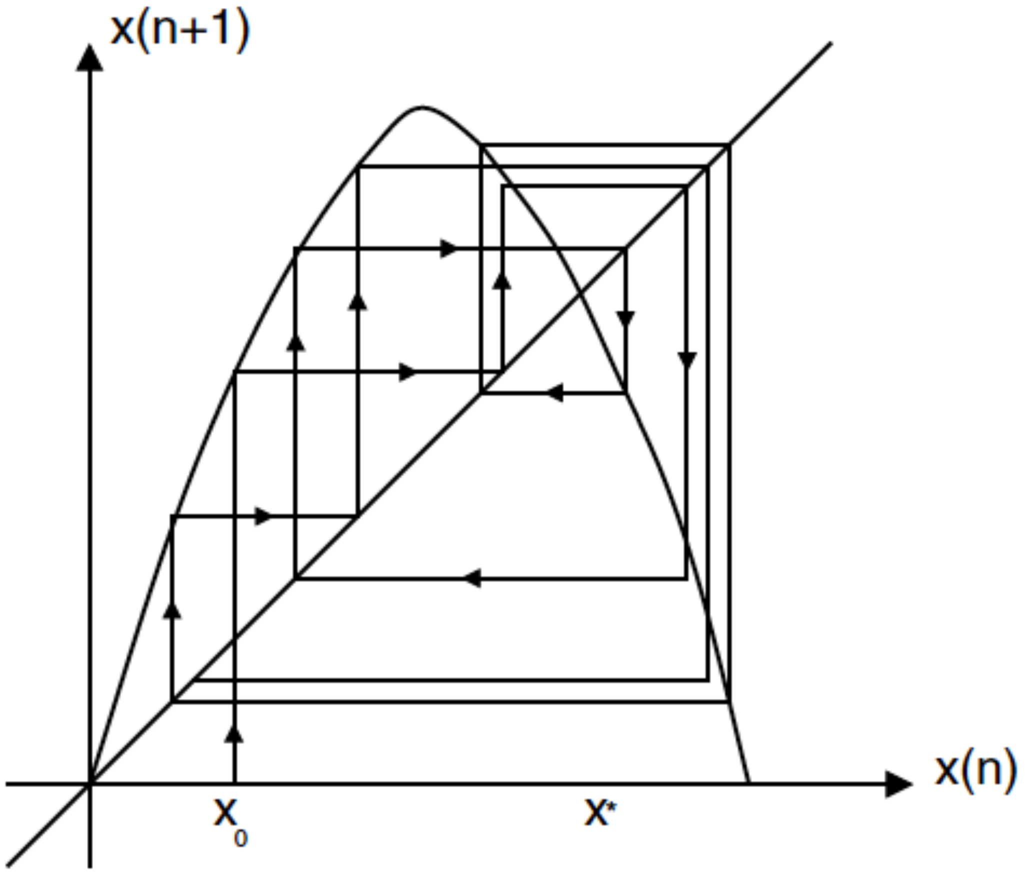


FIGURE 1.33. $\mu > 3$: x^* is an unstable fixed point.

2-cycles

To find the 2-cycles we solve the equation $F_\mu^2(x) = x$ (or we solve $x_2 = \mu x_1(1-x_1), x_1 = \mu x_2(1-x_2)$),

$$\mu^2 x(1-x)[1-\mu x(1-x)] - x = 0. \tag{1.7.3}$$

Discarding the equilibrium points 0 and $x^* = \frac{\mu-1}{\mu}$, one may then divide (1.7.3) by the factor $x(x-(\mu-1)/\mu)$ to obtain the quadratic equation

$$\mu^2 x^2 - \mu(\mu+1)x + \mu + 1 = 0.$$

Solving this equation produces the 2-cycle

$$\begin{aligned} x(0) &= \left[(1+\mu) - \sqrt{(\mu-3)(\mu+1)}\right]/2\mu, \\ x(1) &= \left[(1+\mu) + \sqrt{(\mu-3)(\mu+1)}\right]/2\mu. \end{aligned} \tag{1.7.4}$$

Stability of 2-cycles

2-cycle is asymptotically stable if

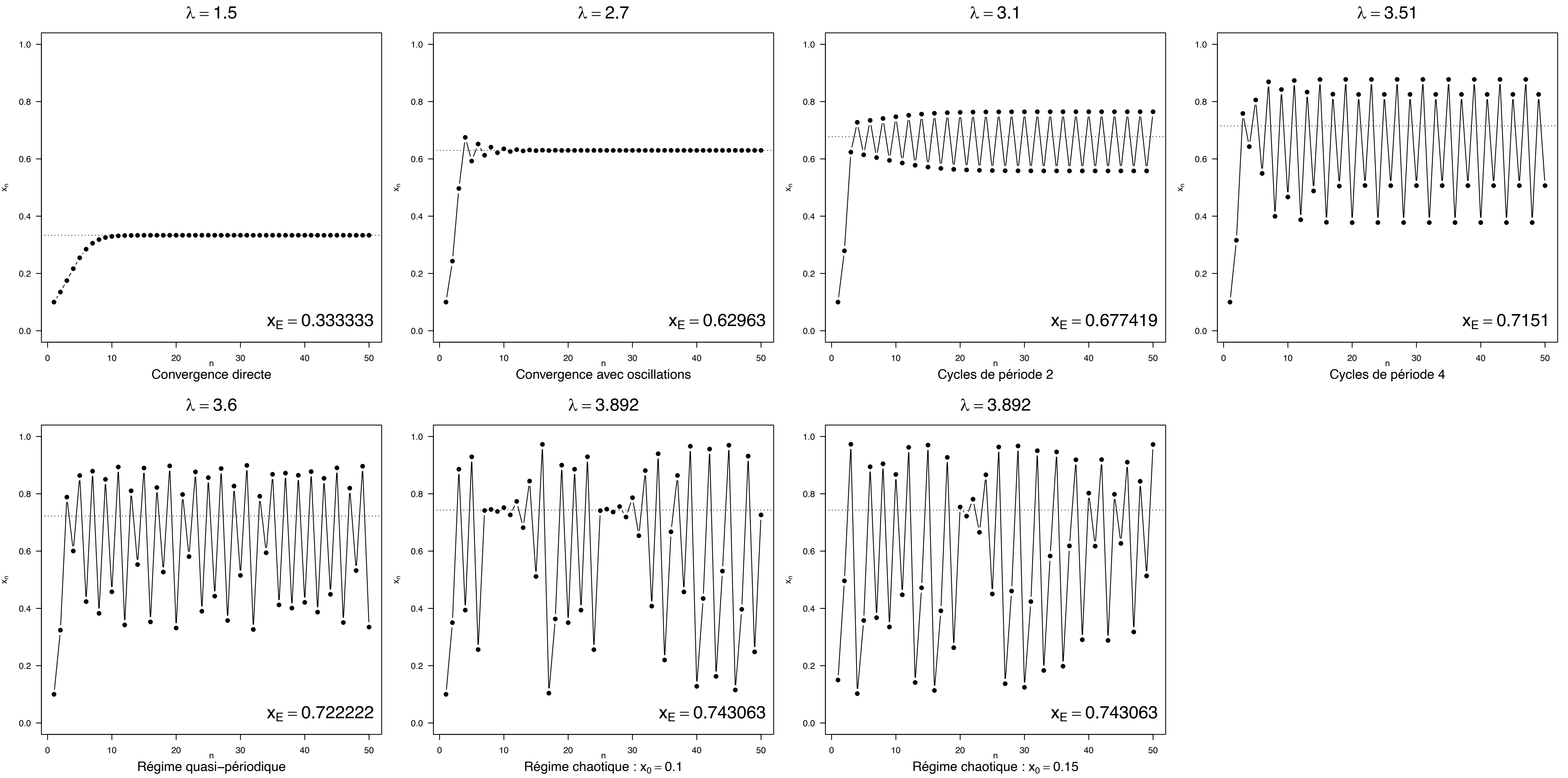
$$|F'_\mu(x(0))F'_\mu(x(1))| < 1,$$

or

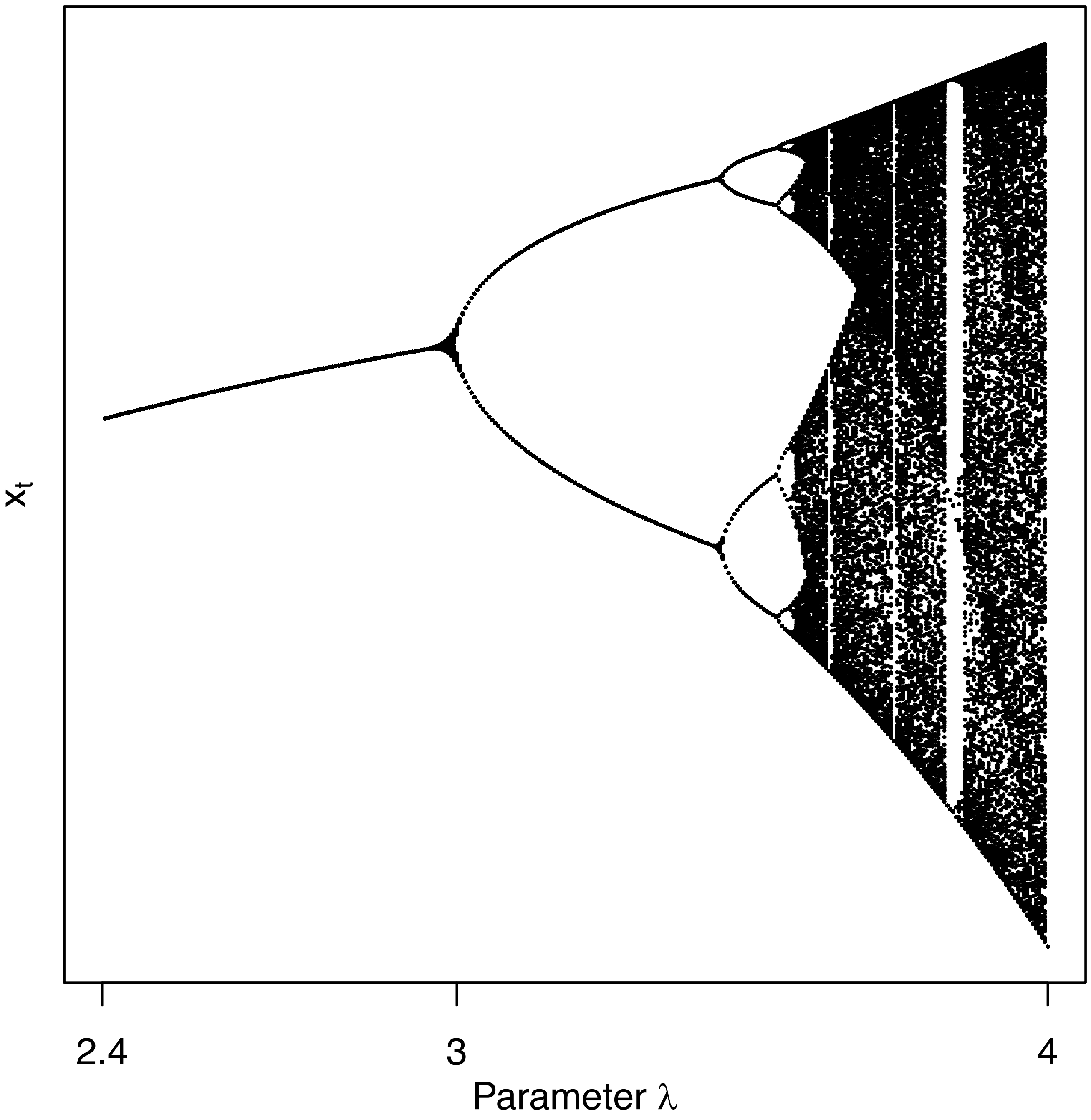
$$-1 < \mu^2(1-2x(0))(1-2x(1)) < 1. \tag{1.7.5}$$

Substituting from (1.7.4) the values of $x(0)$ and $x(1)$ into (1.7.5), we obtain

$$3 < \mu < 1+\sqrt{6} \approx 3.44949.$$



λ = 1.5
x_E = 0.333333
Convergence directe
λ = 2.7
x_E = 0.62963
Convergence avec oscillations
λ = 3.1
x_E = 0.677419
Cycles de période 2
λ = 3.51
x_E = 0.7151
Cycles de période 4
λ = 3.6
x_E = 0.722222
Régime quasi–périodique
λ = 3.892
x_E = 0.743063
Régime chaotique : x_0 = 0.1
λ = 3.892
x_E = 0.743063
Régime chaotique : x_0 = 0.15
x_n
n



x_t
2.4
3
4
Parameter λ

Hassell MP, Lawton JH, May RM. 1976. Patterns of dynamical behavior in single species populations. *J. Anim. Ecol.* 45:471–486.

$$N_{t+1} = \lambda N_t(1+aN_t)^{-\beta}, \qquad (1)$$

where N_t and N_{t+1} are the populations in successive generations, λ is the finite net rate of increase and a and β are constants defining the density dependent feedback term.

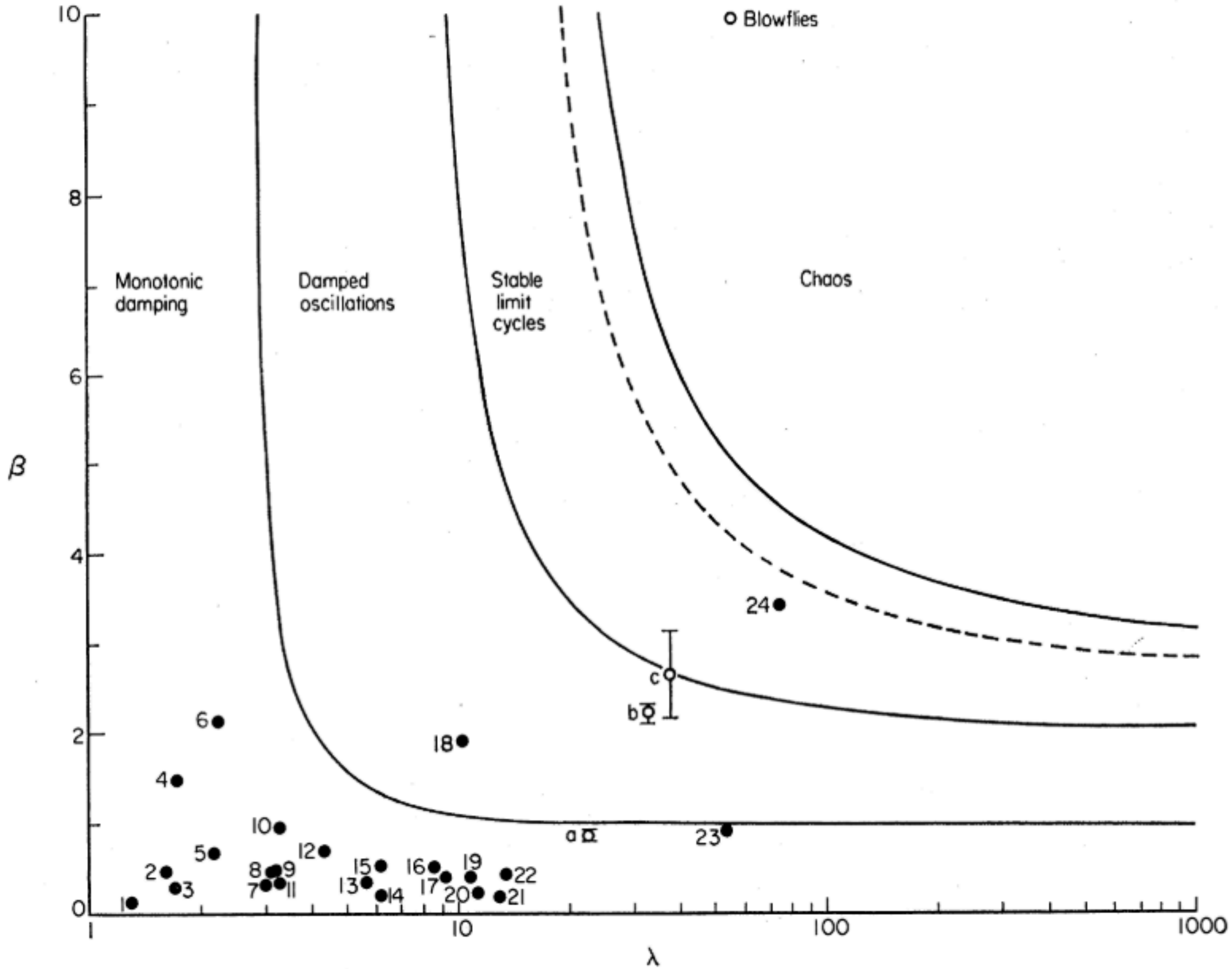


FIG. 2. Stability boundaries between the density dependent parameter, β, and the population growth rate, λ, from eqn (1). The solid lines separate the regions of monotonic and oscillatory damping, stable limit cycles and chaos. The broken line indicates where two-point limit cycles give way to higher order cycles. The solid circles come from the analyses of the life table data in Table 1 and the number by each point refers to this table. The hollow circles are discussed under 'Laboratory experiments'.

Table 1. *Estimates of β and λ, and their 95% confidence limits, from the analysis of insect life table data; numbers correspond to the numbered points in Fig.* 2

No.	Species	β	λ	Author
1	Moth: *Zeiraphera diniana* Gn.	0·1 (0·0–0·2)	1·3 (0·4–4·2)	Auer (1968)
2	Bug: *Anthocoris confusus* (Reuter)	0·5 (0·1–1·3)	1·6 (2·1–8·8)	Evans (1973)
3	Beetle: *Phytodecta olivacea* (Forst.)	0·3 (0·1–0·4)	1·7 (0·9–3·2)	Richards & Waloff (1961)
4	Moth: *Hyphantria cunea* Drury	1·5 (1·4–1·6)	1·7 (0·8–3·8)	Itô, Shibazaki & Iwahashi (1969)
5	Scale: *Parlatoria oleae* (Colvee)	0·7 (−0·3–1·6)	2·2 (1·4–3·3)	Huffaker & Kennett (1966)
6	Bug: *Leptoterna dolobrata* (L.)	2·1 (0·9–3·3)	2·2 (0·9–3·2)	McNeill (1973)
7	Moth: *Erannis defoliaria* (Clerk)	0·4 (0·0–0·7)	3·0 (1·8–5·1)	Ekanayake (1967)
8	Moth: *Bupalus piniarius* L.	0·5 (0·1–0·8)	3·1 (2·1–4·6)	Klomp (1966)
9	Parasitoid fly: *Cyzenis albicans* (F.)	0·5 (0·3–0·7)	3·2 (1·3–8·0)	Hassell (1969)
10	Fly: *Erioischia brassicae* (L.)	1·0 (0·7–1·3)	3·3 (1·2–9·0)	Mukerji (1971)
11	Moth: *Cadra cautella* Walk.	0·3 (0·1–0·6)	3·3 (1·4–7·7)	Benson (1974)
12	Bug: *Nezara viridula* L.	0·7 (0·1–1·3)	4·3 (2·1–8·8)	Kiritani, Hokyo & Kimura (1967)
13	Moth: *Operophtera brumata* (L.)	0·3 (0·2–0·5)	5·5 (3·2–9·3)	Varley & Gradwell (1968)
14	Bug: *Nephotettix cincticeps* Uhler	0·2 (0·1–0·3)	6·1 (3·6–10·4)	Kiritani *et al.* (1970)
15	Moth: *Erannis progemmaria* (Hb.)	0·5 (0·1–1·0)	6·3 (4·0–10·0)	Ekanayake (1967)
16	Moth: *Anagasta kuehniella* (Zell.)	0·5 (0·3–0·7)	8·6 (7·3–10·1)	Hassell & Huffaker (1969)
17	Bug: *Neophilaenus lineatus* (L.)	0·4 (0·3–0·5)	9·2 (7·5–11·4)	Whittaker (1971)
18	Mosquito: *Aedes aegypti* (L.)	1·9 (0·7–3·1)	10·6 (6·4–17·5)	Southwood *et al.* (1972)
19	Moth: *Tyria jacobaeae* L.	0·4 (0·1–0·7)	10·7 (1·6–72·4)	Dempster (1975)
20	Moth: *Erannis leucophaearia* (Schiff.)	0·2 (0·0–0·5)	11·2 (7·6–16·6)	Ekanayake (1967)
21	Moth: *Acleris variana* Fern.	0·2 (0·0–0·4)	13·0 (6·2–27·1)	Morris (1959)
22	Bug: *Saccarosydne saccharivora* (Ww.)	0·4 (0·1–0·7)	13·5 (7·8–23·5)	Metcalfe (1972)
23	Parasitoid wasp: *Bracon hebetor* Say	0·9 (0·4–1·4)	54·0 (27·1–107·8)	Benson (1974)
24	Beetle: *Leptinotarsa decemlineata* (Say)	3·4 (2·5–4·3)	75·0 (44·2–127·3)	Harcourt (1971)