English support for the course on difference equations (2)

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For all details, refer to Elaydi S. 2005.

An introduction to difference equations

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https://filesender.renater.fr/?s=download&token=4e60d8ad-8d46-79eb-697a-07c2571e6da8

Reminder on linear algebra

- Let **A** and **B** two k x k matrices. They are **similar** if there exists a nonsingular matrix **P** such that $\mathbf{P^{-1}AP} = \mathbf{B}$.
- It may be shown in this case that **A** and **B** have the same eigenvalues.
- If a matrix **A** is similar to a diagonal matrix $\mathbf{D} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k)$, then **A** is said to be *diagonalizable*. Notice here that the diagonal elements of **D**, namely $\lambda_1, \lambda_2, \ldots, \lambda_k$, are the eigenvalues of **A**.
- For diagonalizable matrices, computing $\mathbf{A^n}$ is simple: $\mathbf{A^n} = \mathbf{PD^nP^{-1}}$, with $\mathbf{D^n} = \mathrm{diag}(\lambda_1^{\,n}, \lambda_2^{\,n}, \ldots, \lambda_k^{\,n})$.

Theorem *A $k \times k$ matrix is diagonalizable if and only if it has k linearly independent eigenvectors.*

Reminder on Jordan's forms

- General case where the matrix $\mathbf{A}$ is not diagonalizable; this may happen when $\mathbf{A}$ has repeated eigenvalues, or is not able to generate k linearly independent eigenvectors.
- In dimension 2, $\mathbf{J} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, with $\mathbf{J}$ of one of the following forms:

$$(a) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \lambda_1 \neq \lambda_2 \qquad (b) \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix}$$

$$(c) \begin{pmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{pmatrix} \qquad (d) \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \quad \beta > 0$$

Reminder on Jordan's forms (*continued*)

- The Jordan's form depends on eigenvalues of matrix $\mathbf{A}$:
- $\det(A - \lambda\, I) = 0 \Leftrightarrow \lambda^2 - \mathrm{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \begin{aligned} \mathrm{tr}(\mathbf{A}) &= a_{11} + a_{22} \\ \det(\mathbf{A}) &= a_{11}a_{22} - a_{12}a_{21} \end{aligned}$$

- Example:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -2 & 4 \end{pmatrix}$$

Proposal

Any recursive system $\mathrm{X_{n+1}} = \mathbf{A}\ \mathrm{X_n}$ can be transformed in an equivalent canonical recursive system $\mathrm{Y_{n+1}} = \mathbf{J}\ \mathrm{Y_n}$, where $\mathbf{J} = \mathbf{P^{-1}}\ \mathbf{A}\ \mathbf{P}$ is the Jordan's form associated to $\mathbf{A}$ and $\mathrm{X_n} = \mathbf{P}\ \mathrm{Y_n}$.

Let $\mathrm{X_n} = (\mathrm{x_n}, \mathrm{y_n})$ and $\mathrm{Y_n} = (\mathrm{w_n}, \mathrm{z_n})$

$\mathbf{J} = \mathbf{P^{-1}}\ \mathbf{A}\ \mathbf{P} \Leftrightarrow \mathbf{A} = \mathbf{P}\ \mathbf{J}\ \mathbf{P^{-1}}$ and $\mathbf{A^n} = \mathbf{P}\ \mathbf{J^n}\ \mathbf{P^{-1}}$.

Solve a linear recursive system means calculate $\mathbf{J^n}$

This calculation depends on the type of eigenvalues.

If A has 2 distinct real eigenvalues

$$\begin{pmatrix} w_{n+1} \\ z_{n+1} \end{pmatrix} = \mathbf{J} \begin{pmatrix} w_n \\ z_n \end{pmatrix} \text{ avec } \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \mathbf{P} \begin{pmatrix} w_n \\ z_n \end{pmatrix}$$

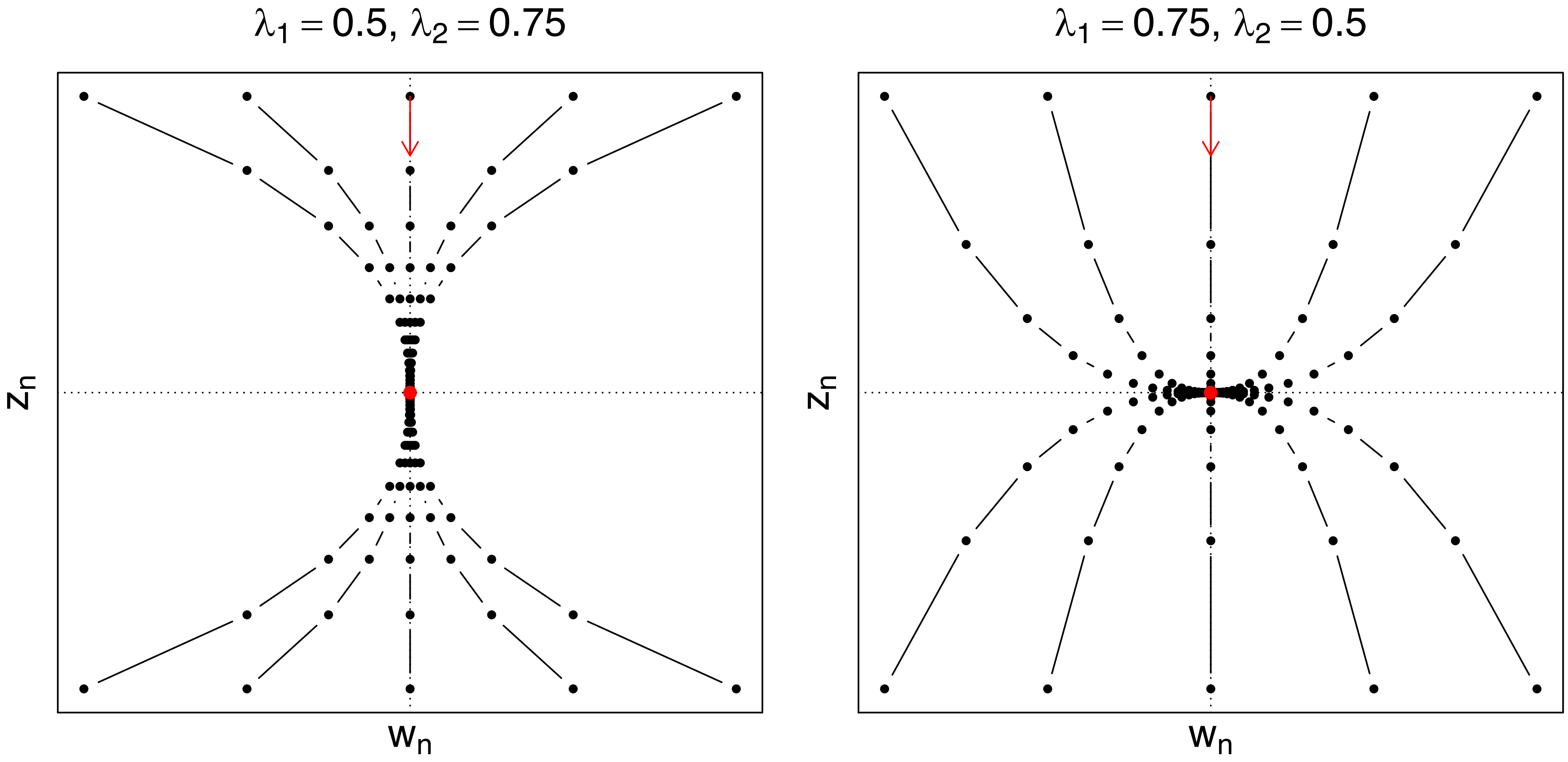
$$\begin{pmatrix} w_n \\ z_n \end{pmatrix} = \mathbf{J}^n \begin{pmatrix} w_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} w_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} \lambda_1^n w_0 \\ \lambda_2^n z_0 \end{pmatrix}$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \mathbf{P} \begin{pmatrix} w_n \\ z_n \end{pmatrix} = \left(\begin{array}{c|c} v_{11} & v_{21} \\ v_{12} & v_{22} \end{array}\right) \begin{pmatrix} \lambda_1^n w_0 \\ \lambda_2^n z_0 \end{pmatrix}$$

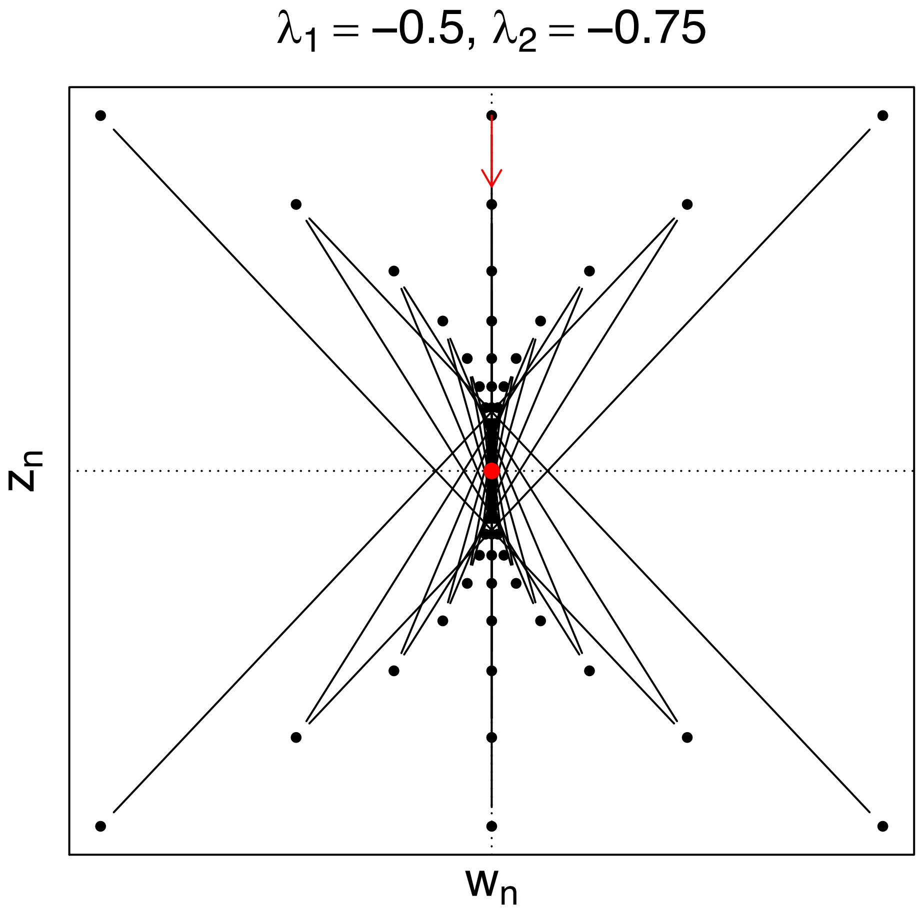
$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} v_{11}\lambda_1^n w_0 + v_{21}\lambda_2^n z_0 \\ v_{12}\lambda_1^n w_0 + v_{22}\lambda_2^n z_0 \end{pmatrix}$$

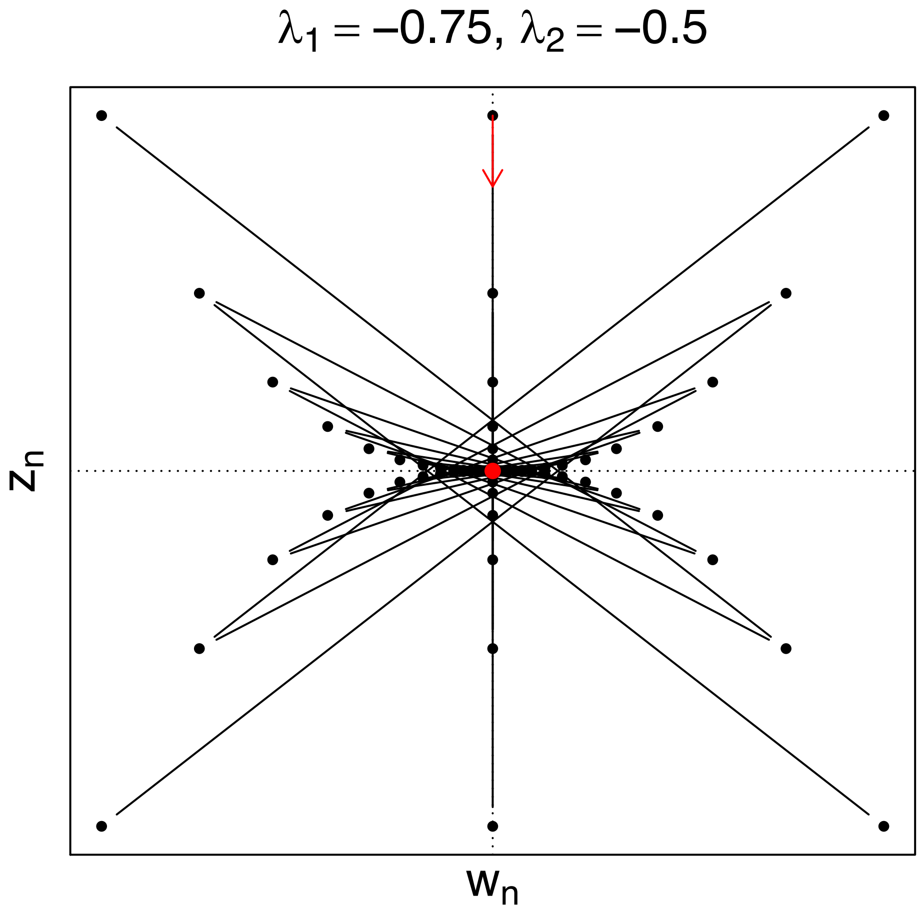
$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = w_0 \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} \lambda_1^n + z_0 \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} \lambda_2^n$$

If **A** has 2 distinct real eigenvalues (*continued*)

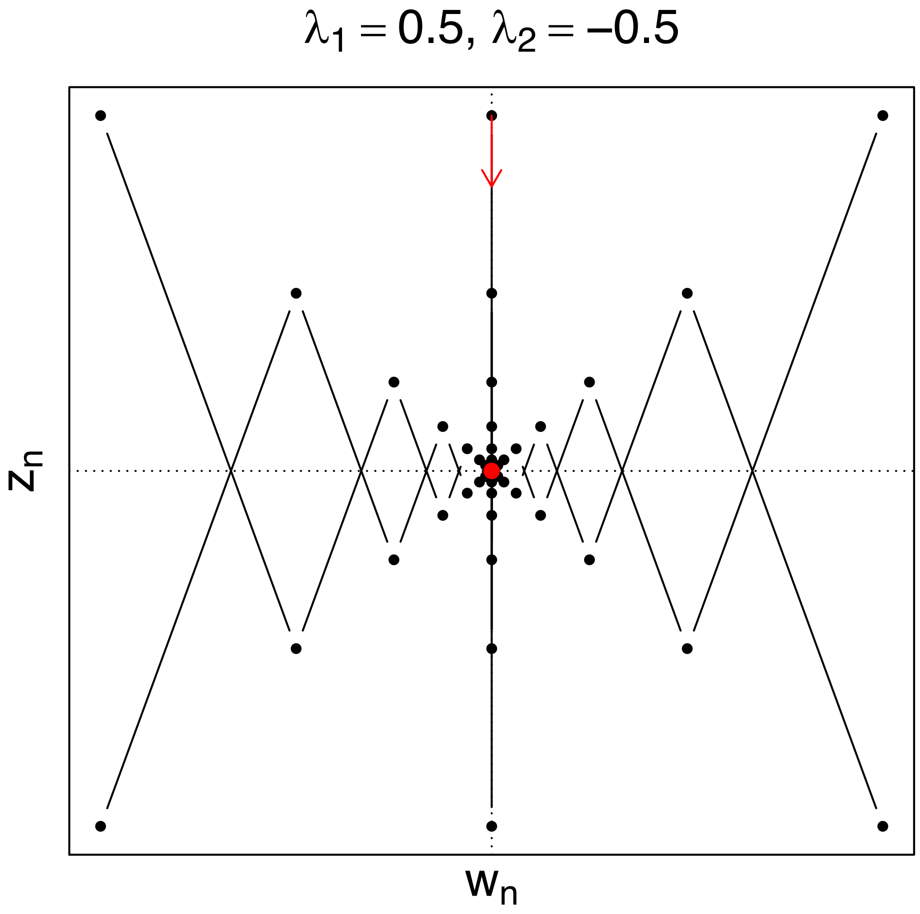


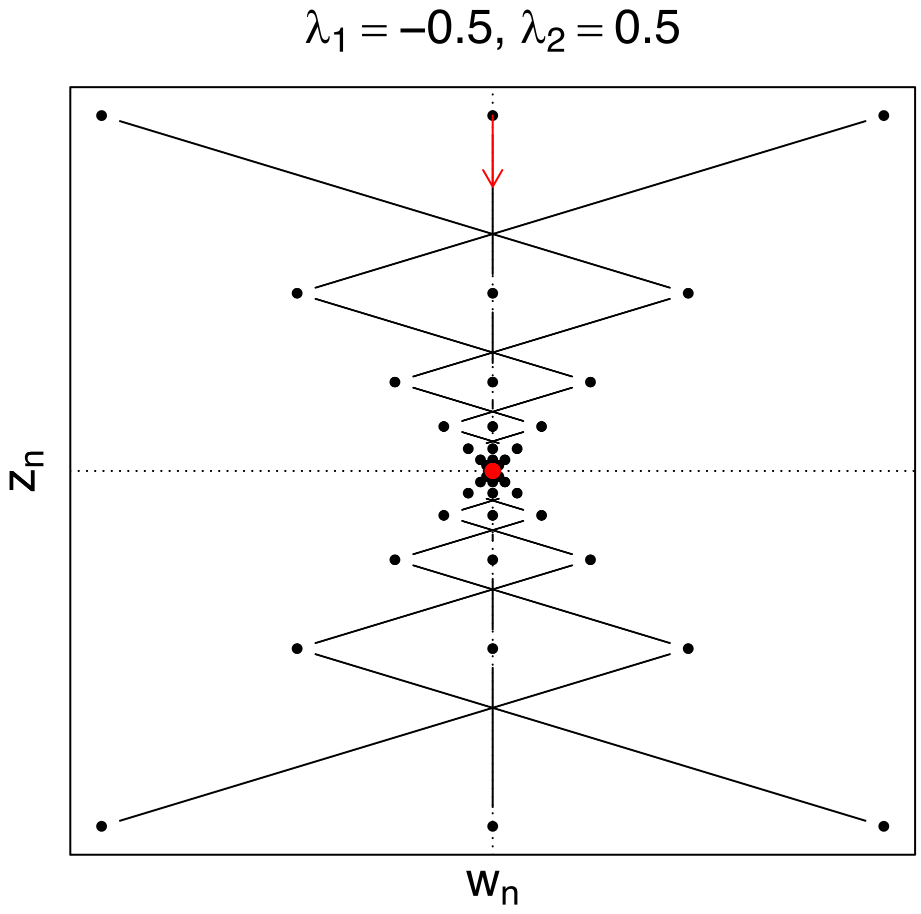
If **A** has 2 distinct real eigenvalues (*continued*)



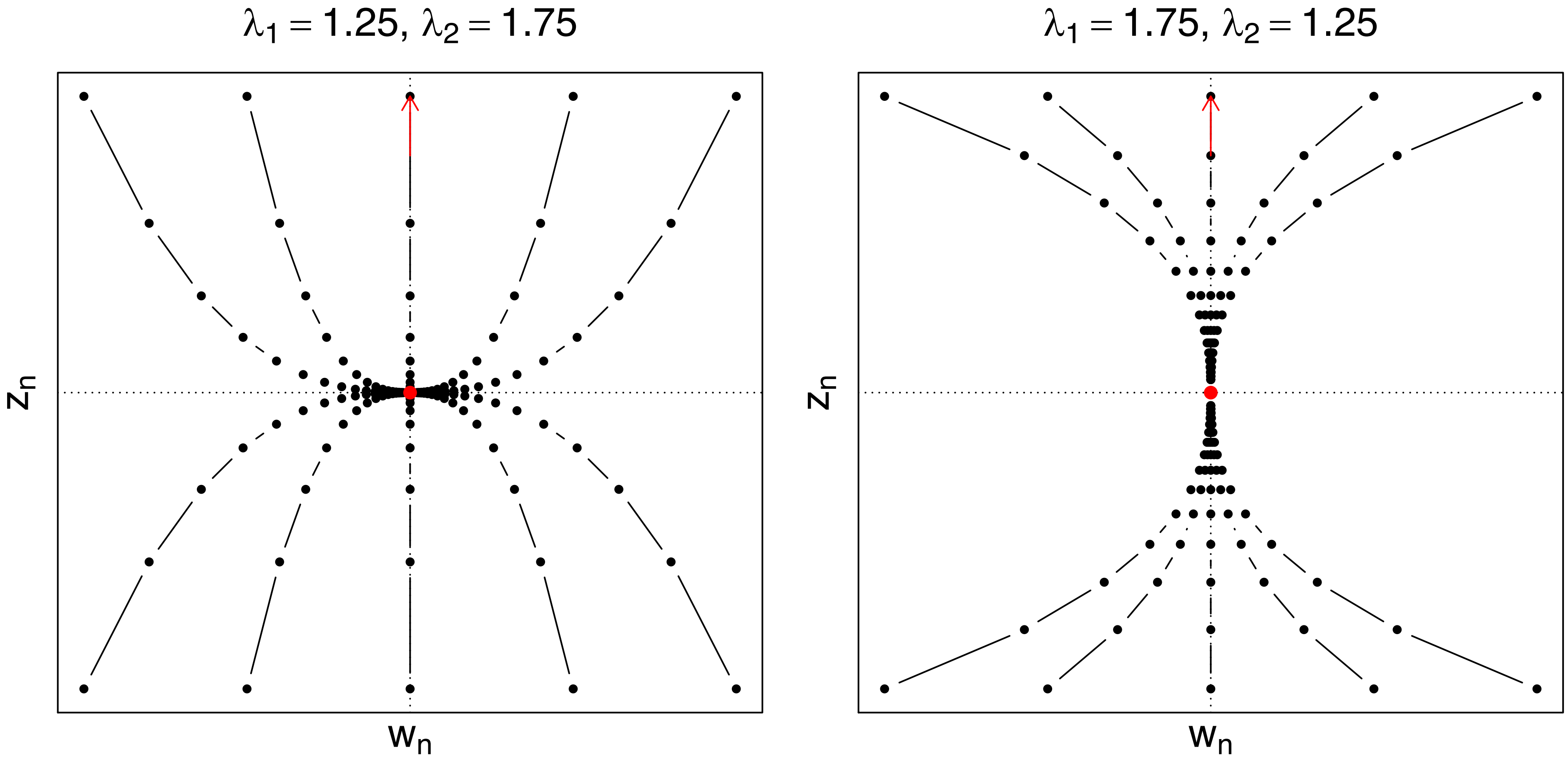


If **A** has 2 distinct real eigenvalues (*continued*)

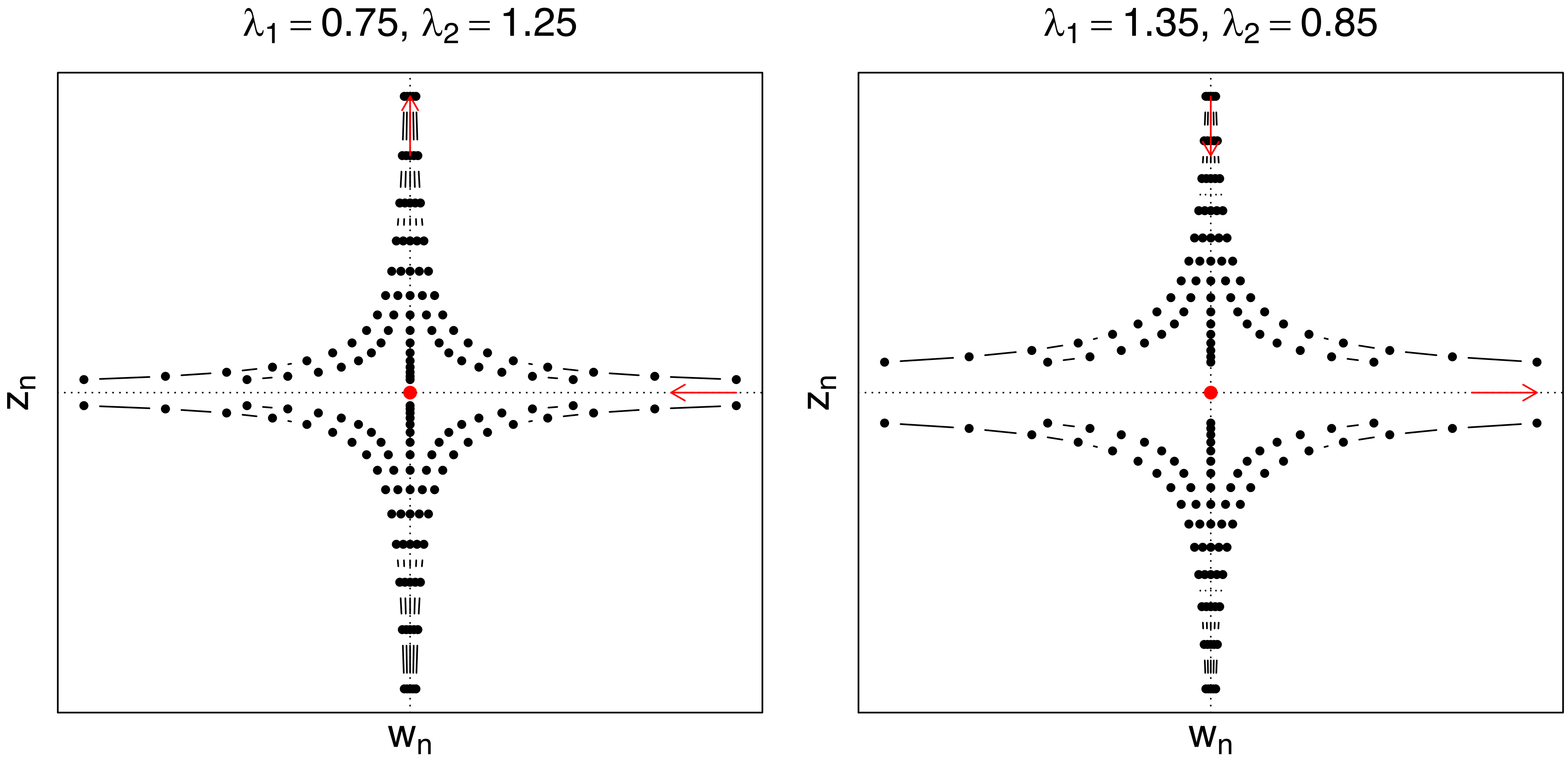




If **A** has 2 distinct real eigenvalues (*continued*)

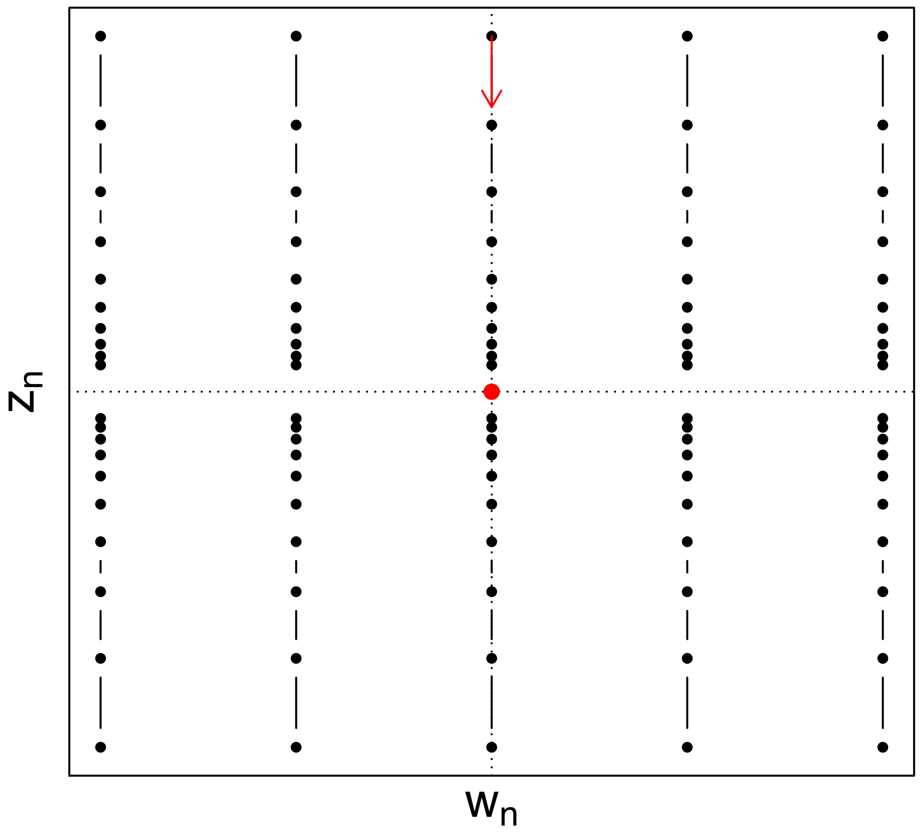


If A has 2 distinct real eigenvalues (*continued*)

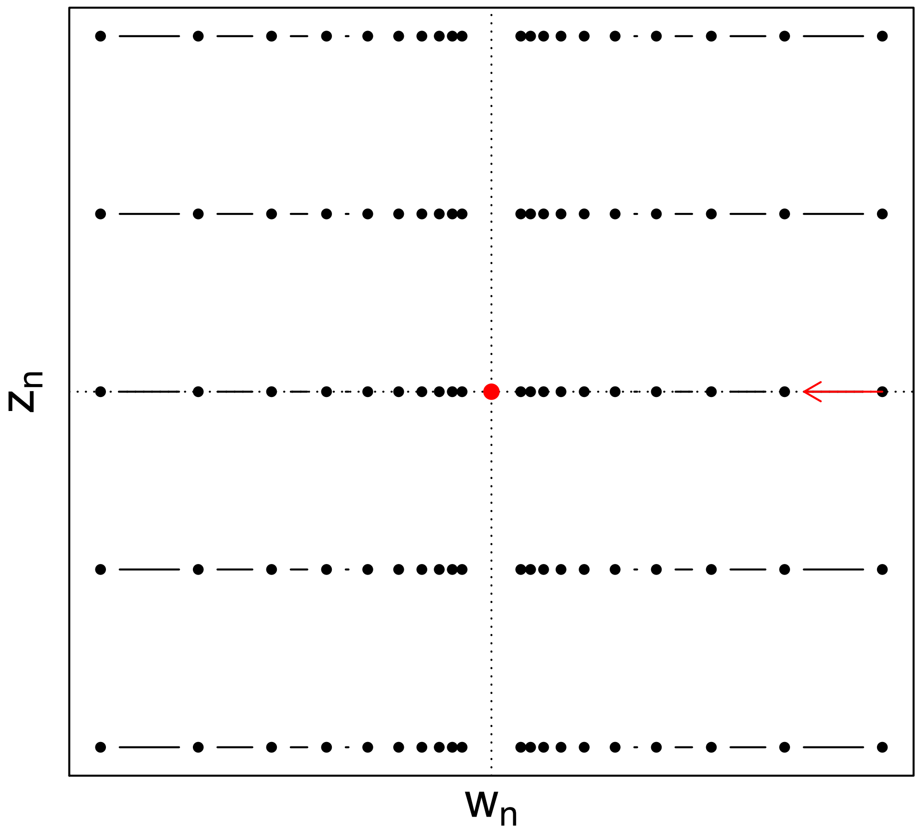


If **A** has 2 distinct real eigenvalues (*continued*)

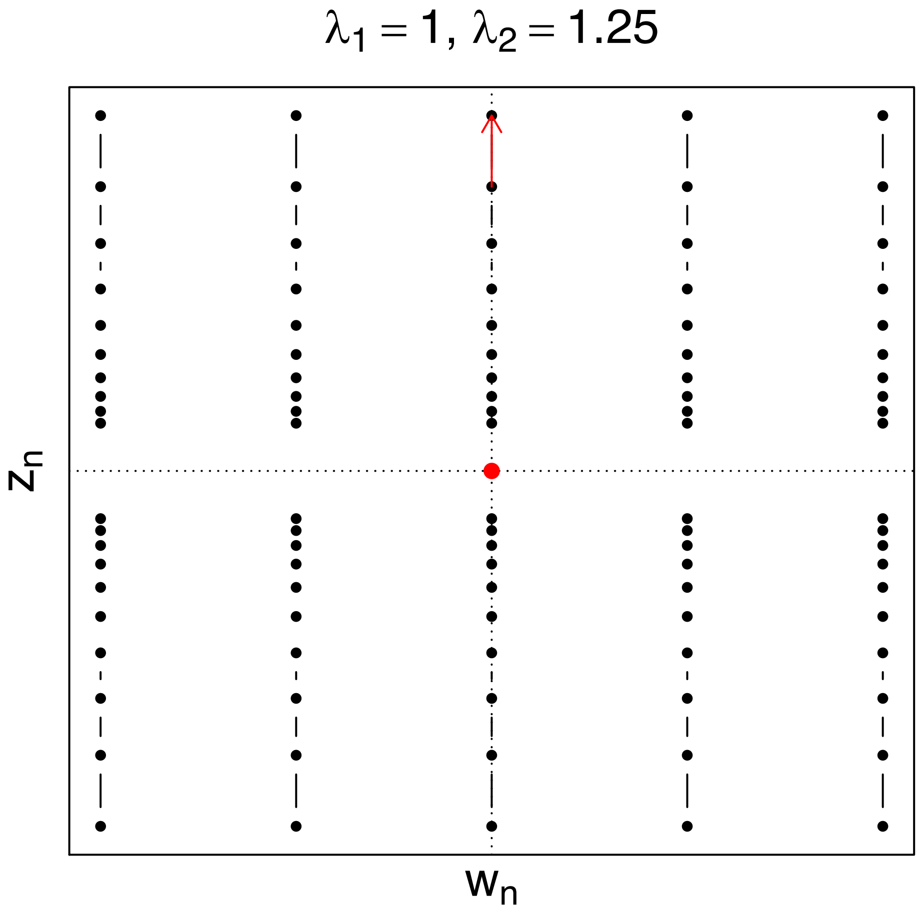
$\lambda_1 = 1,\ \lambda_2 = 0.75$

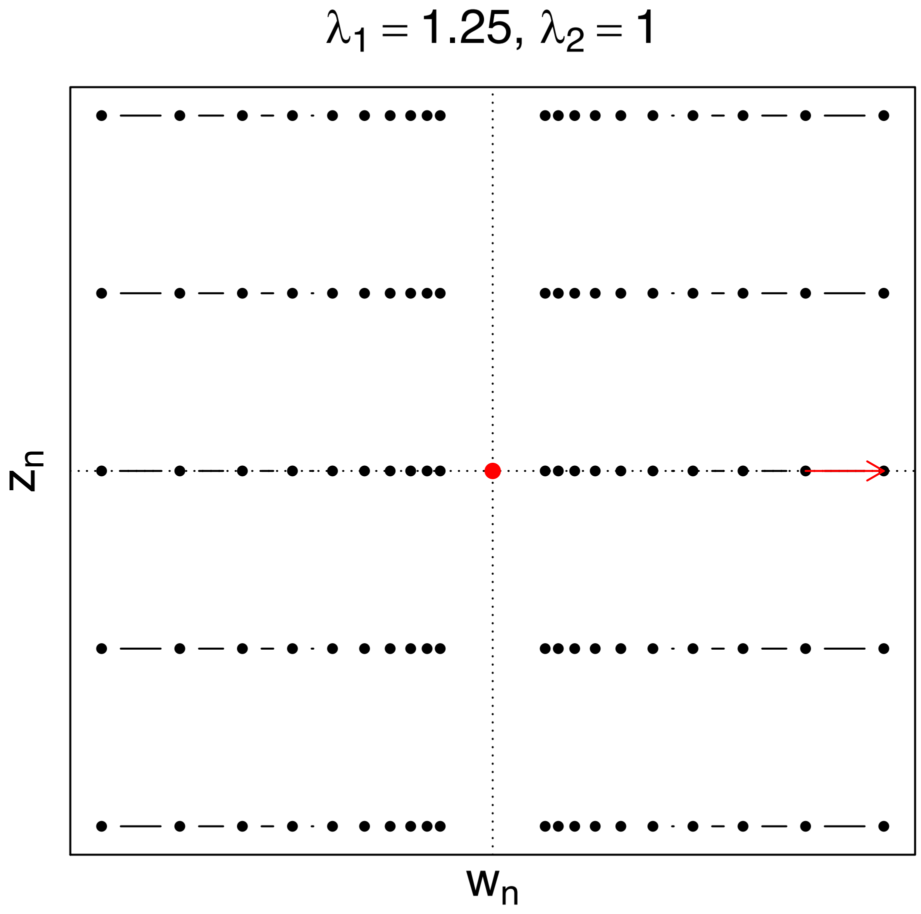


$\lambda_1 = 0.75,\ \lambda_2 = 1$



If A has 2 distinct real eigenvalues (*continued*)





If A has 2 one double eigenvalue

$$\begin{pmatrix} w_{n+1} \\ z_{n+1} \end{pmatrix} = \mathbf{J} \begin{pmatrix} w_n \\ z_n \end{pmatrix} \text{avec} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \mathbf{P} \begin{pmatrix} w_n \\ z_n \end{pmatrix}$$

$$\begin{pmatrix} w_n \\ z_n \end{pmatrix} = \mathbf{J^n} \begin{pmatrix} w_0 \\ z_0 \end{pmatrix}$$

$$\mathbf{J^2} = \begin{pmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{pmatrix} \begin{pmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{pmatrix} = \begin{pmatrix} \lambda_0^2 & 2\lambda_0 \\ 0 & \lambda_0^2 \end{pmatrix}$$

$$\mathbf{J^3} = \begin{pmatrix} \lambda_0^2 & 2\lambda_0 \\ 0 & \lambda_0^2 \end{pmatrix} \begin{pmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{pmatrix} = \begin{pmatrix} \lambda_0^3 & 3\lambda_0^2 \\ 0 & \lambda_0^3 \end{pmatrix}$$

$$\vdots$$

$$\mathbf{J^n} = \begin{pmatrix} \lambda_0^n & n\lambda_0^{n-1} \\ 0 & \lambda_0^n \end{pmatrix}$$

$$\begin{pmatrix} w_n \\ z_n \end{pmatrix} = \begin{pmatrix} \lambda_0^n & n\lambda_0^{n-1} \\ 0 & \lambda_0^n \end{pmatrix} \begin{pmatrix} w_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} \lambda_0^n w_0 + n\lambda_0^{n-1} z_0 \\ \lambda_0^n z_0 \end{pmatrix}$$

If A has 2 one double eigenvalue (*continued*)

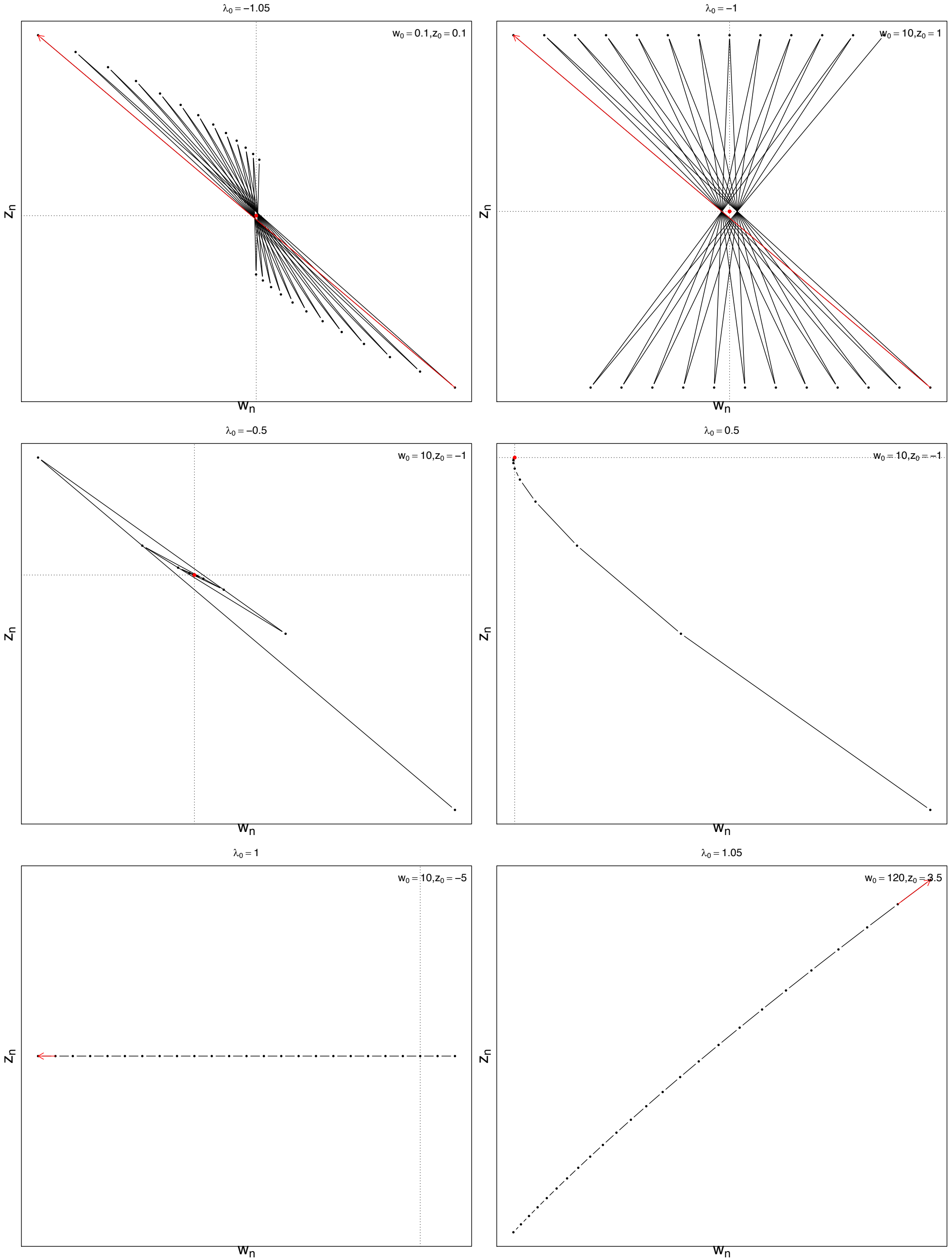
$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} v_{01} & m_{21} \\ v_{02} & m_{22} \end{pmatrix} \begin{pmatrix} \lambda_0^n w_0 + n\lambda_0^{n-1} z_0 \\ \lambda_0^n z_0 \end{pmatrix}$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} v_{01}\left(\lambda_0^n w_0 + n\lambda_0^{n-1} z_0\right) + m_{21}\lambda_0^n z_0 \\ v_{02}\left(\lambda_0^n w_0 + n\lambda_0^{n-1} z_0\right) + m_{22}\lambda_0^n z_0 \end{pmatrix}$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = (\lambda_0 w_0 + n z_0) \begin{pmatrix} v_{01} \\ v_{02} \end{pmatrix} \lambda_0^{n-1} + z_0 \begin{pmatrix} m_{21} \\ m_{22} \end{pmatrix} \lambda_0^n$$

$$\lim_{n \to +\infty} n\lambda_0^{n-1} = 0 \text{ si } |\lambda_0| < 1.$$

If **A** has 2 one double eigenvalue (*continued*)



If A has 2 conjugate eigenvalues

$$\mathbf{J}=\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}=\begin{pmatrix}\rho\cos\omega & -\rho\sin\omega\\ \rho\sin\omega & \rho\cos\omega\end{pmatrix}=\rho R(\omega)$$

$$\begin{aligned}
&\mathbf{J}=\rho R(\omega)\\
&\mathbf{J}^2=\rho R(\omega)\,\rho R(\omega)=\rho^2 R(2\omega)\\
&\vdots\\
&\mathbf{J}^n=\rho^n R(n\omega)
\end{aligned}$$

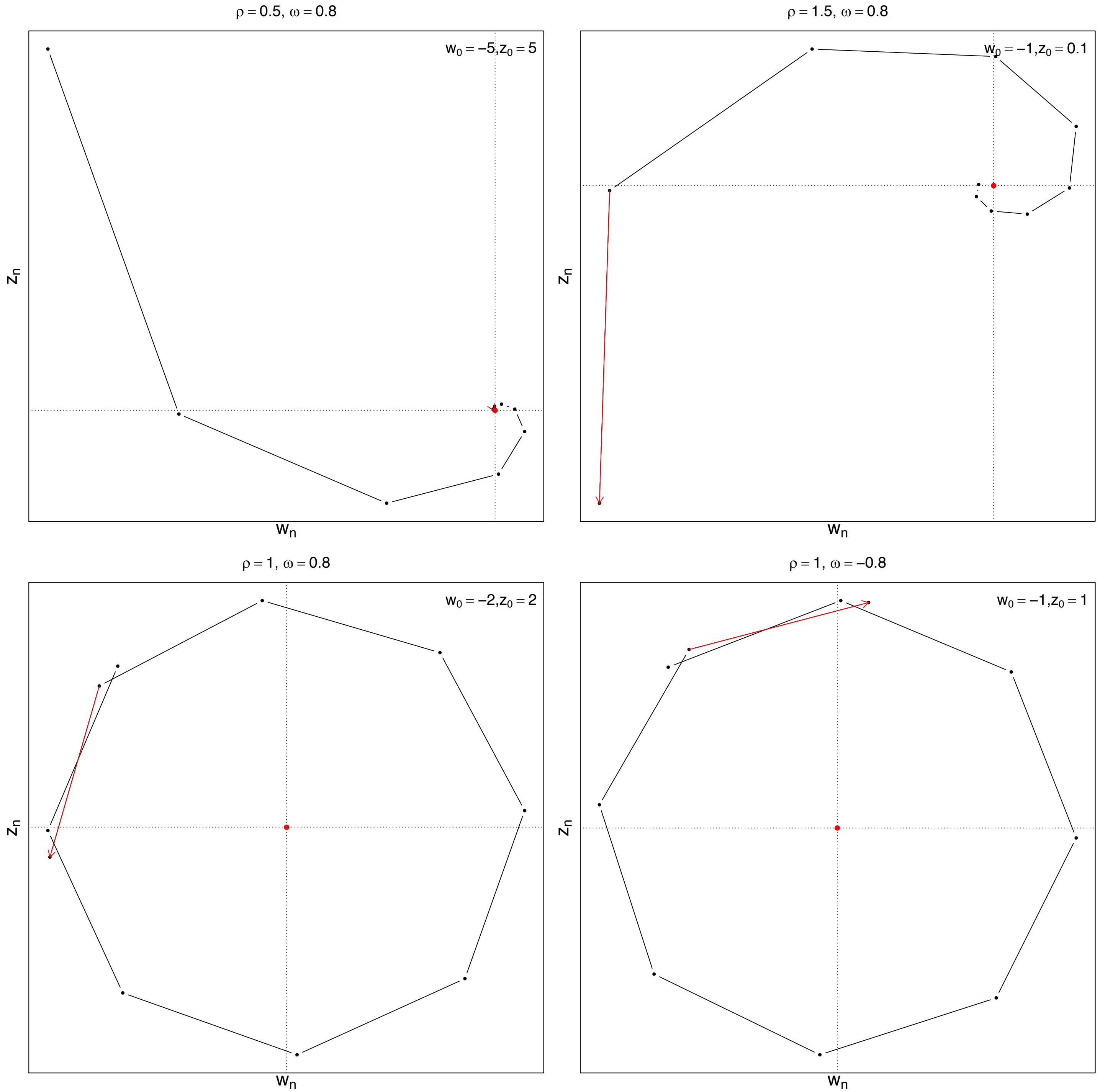
$$\boxed{\begin{aligned}
\begin{pmatrix}x_n\\ y_n\end{pmatrix}&=\begin{pmatrix}\vec{b} & \vec{a}\end{pmatrix}\rho^n\begin{pmatrix}\cos n\omega w_0-\sin n\omega z_0\\ \sin n\omega w_0+\cos n\omega z_0\end{pmatrix}\\
\begin{pmatrix}x_n\\ y_n\end{pmatrix}&=\rho^n\left[(\cos n\omega w_0-\sin n\omega z_0)\,\vec{b}+(\sin n\omega w_0+\cos n\omega z_0)\,\vec{a}\right]
\end{aligned}}$$

$$\begin{pmatrix}w_n\\ z_n\end{pmatrix}=\rho^n R(n\omega)\begin{pmatrix}w_0\\ z_0\end{pmatrix}=\rho^n\begin{pmatrix}\cos n\omega & -\sin n\omega\\ \sin n\omega & \cos n\omega\end{pmatrix}\begin{pmatrix}w_0\\ z_0\end{pmatrix}=\rho^n\begin{pmatrix}\cos n\omega w_0-\sin n\omega z_0\\ \sin n\omega w_0+\cos n\omega z_0\end{pmatrix}$$

If A has 2 conjugate eigenvalues (*continued*)

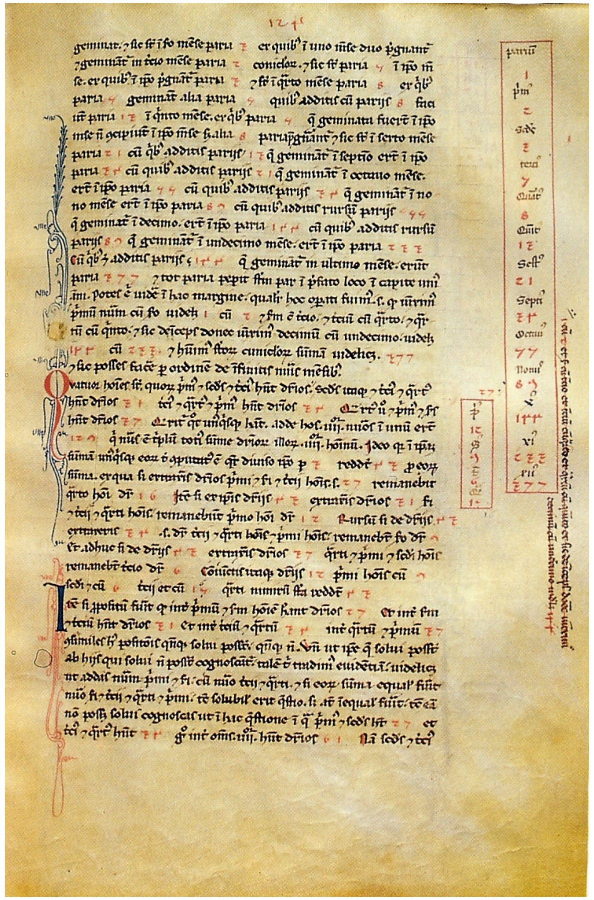
Example

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}$$



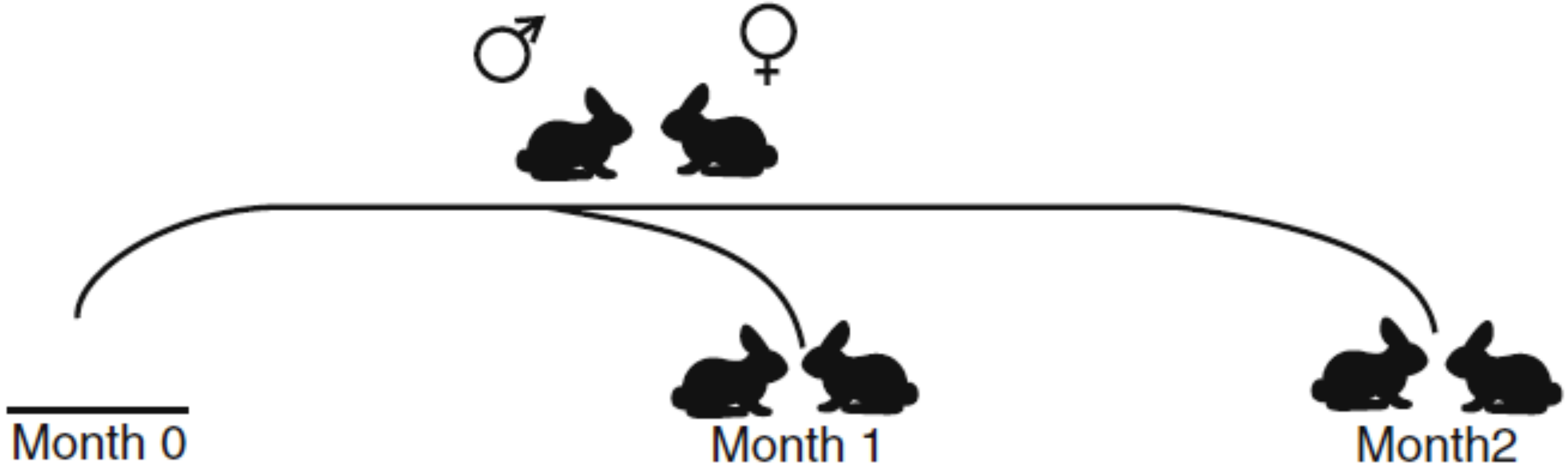
The Fibonacci Sequence (The Rabbit Problem)

- This problem first appeared in 1202, in *Liber abaci*, a book about the abacus, written by the famous Italian mathematician **Leonardo di Pisa**, better known as **Fibonacci**.



The Fibonacci Sequence (The Rabbit Problem)

- This problem first appeared in 1202, in *Liber abaci*, a book about the abacus, written by the famous Italian mathematician **Leonardo di Pisa**, better known as **Fibonacci**.
- The problem may be stated as follows: How many pairs of rabbits will there be after one year if starting with one pair of mature rabbits, if each pair of rabbits gives birth to a new pair each month starting when it reaches its maturity age of two months?



The Fibonacci Sequence (The Rabbit Problem)

TABLE 2.2. Rabbits' population size.

Month	0	1	2	3	4	5	6	7	8	9	10	11	12
Pairs	1	2	3	5	8	13	21	34	55	89	144	233	377

The first pair has offspring at the end of the first month, and thus we have two pairs. At the end of the second month only the first pair has offspring, and thus we have three pairs. At the end of the third month, the first and second pairs will have offspring, and hence we have five pairs. Continuing this procedure, we arrive at Table 2.2. If $F(n)$ is the number of pairs of rabbits at the end of n months, then the recurrence relation that represents this model is given by the second-order linear difference equation

$$F(n+2) = F(n+1) + F(n), \qquad F(0) = 1, \qquad F(1) = 2, \qquad 0 \leq n \leq 10.$$

This example is a special case of the Fibonacci sequence, given by

$$F(n+2) = F(n+1) + F(n), \qquad F(0) = 0, \qquad F(1) = 1, \qquad n \geq 0.$$

The Fibonacci Sequence as a linear system

$$\begin{pmatrix} j_{n+1} \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} a_n \\ j_n + a_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} j_n \\ a_n \end{pmatrix} \qquad j_0 = 1 \qquad a_0 = 0.$$

$$a_{n+2} = j_{n+1} + a_{n+1} = a_n + a_{n+1}.$$

$$\lambda^2 - \lambda - 1 = 0 \qquad \lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}, \qquad \mathbf{P} = \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix}$$

$$\mathbf{J} = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \qquad \mathbf{P}^{-1} = -\frac{1}{\sqrt{5}} \begin{pmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ -\frac{1+\sqrt{5}}{2} & 1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -\left(\frac{1-\sqrt{5}}{2}\right) & 1 \\ \frac{1+\sqrt{5}}{2} & -1 \end{pmatrix}$$

The Fibonacci Sequence as a linear system (*continued*)

$$\begin{pmatrix} j_n \\ a_n \end{pmatrix} = w_0 \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n + z_0 \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\begin{pmatrix} w_0 \\ z_0 \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} j_0 \\ a_0 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -\left(\frac{1-\sqrt{5}}{2}\right) & 1 \\ \frac{1+\sqrt{5}}{2} & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right) \\ \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right) \end{pmatrix}$$

$$a_n = w_0 \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + z_0 \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

$$a_n = -\frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right) \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right) \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

$$a_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right)$$

$$n \to +\infty,\ a_n \simeq \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n, \qquad \lim_{n \to +\infty} \left(\frac{1-\sqrt{5}}{2}\right)^n = 0.$$

$$\frac{a_{n+1}}{a_n} \simeq \frac{1+\sqrt{5}}{2}.$$

The golden number

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots.$$

- If $a > b > 0$, then $\frac{a+b}{a} = \frac{a}{b} \stackrel{\text{def}}{=} \varphi$, $1 + \frac{1}{\varphi} = \varphi$. $\varphi + 1 = \varphi^2$

- Named ϕ to pay homage to the Greek sculptor Phidias (490-430 BC) who decorated the Parthenon in Athena with many gold rectangles.

Lindenmayer systems (or L-systems)

- Introduced in 1968 by **Aristid Lindenmayer**, a Hungarian theoretical biologist and botanist at the University of Utrecht.

- L-systems consist of:
  - an alphabet of symbols that can be used to make strings;
  - a collection of production rules that expand each symbol into some larger string of symbols;
  - an initial “axiom” string from which to begin construction;
  - and a mechanism for translating the generated strings into geometric structures.

- Lindenmayer A. 1968. Mathematical models for cellular interactions in development I. Filaments with one-sided inputs. *J. Theor. Biol.* 18:280–299.

A very simple L-system in 1D

- Consider cells in two categories:
  1. Young an immature cells, denoted by ***a***, that do not divide;
  2. Mature cells, denoted by ***b***, that divide.

- Reproduction is discrete and from one step t to the next $t+1$:
  - Immature cells become mature ones: $a \rightarrow b$
  - Mature cells reproduce in one cell a and one cell b: $b \rightarrow ab$

- Starting from a unique cell a:

  $a \rightarrow b \rightarrow ab \rightarrow bab \rightarrow abbab \rightarrow bababbab \rightarrow \ldots$

- Let $N_a(t)$ and $N_b(t)$ be numbers of cells a and b at time t.

A very simple L-system in 1D (*continued*)

$$\begin{cases} N_a(t+1) = N_b(t) \\ N_b(t+1) = N_a(t) + N_b(t) \end{cases} \Leftrightarrow \begin{pmatrix} N_a(t+1) \\ N_b(t+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} N_a(t) \\ N_b(t) \end{pmatrix}$$

$$\begin{pmatrix} N_a(t) \\ N_b(t) \end{pmatrix} = -\frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right) \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^t + \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right) \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix} \left(\frac{1-\sqrt{5}}{2}\right)^t$$

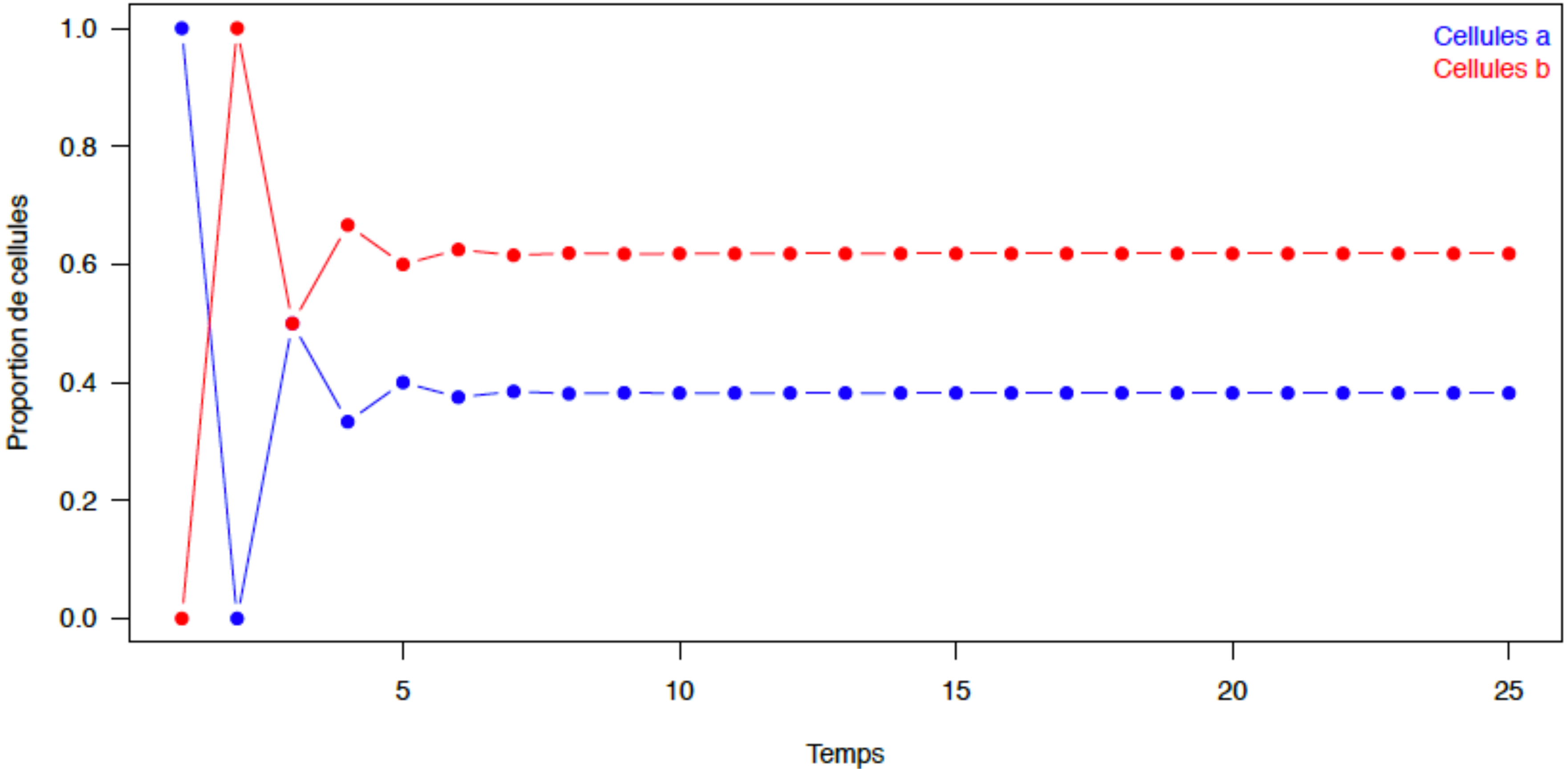
$$\lim_{t \to +\infty} \left(\tfrac{1-\sqrt{5}}{2}\right)^t = 0$$

$$\begin{pmatrix} N_a(t) \\ N_b(t) \end{pmatrix} \simeq -\frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right) \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^t$$

A very simple L-system in 1D (*continued*)

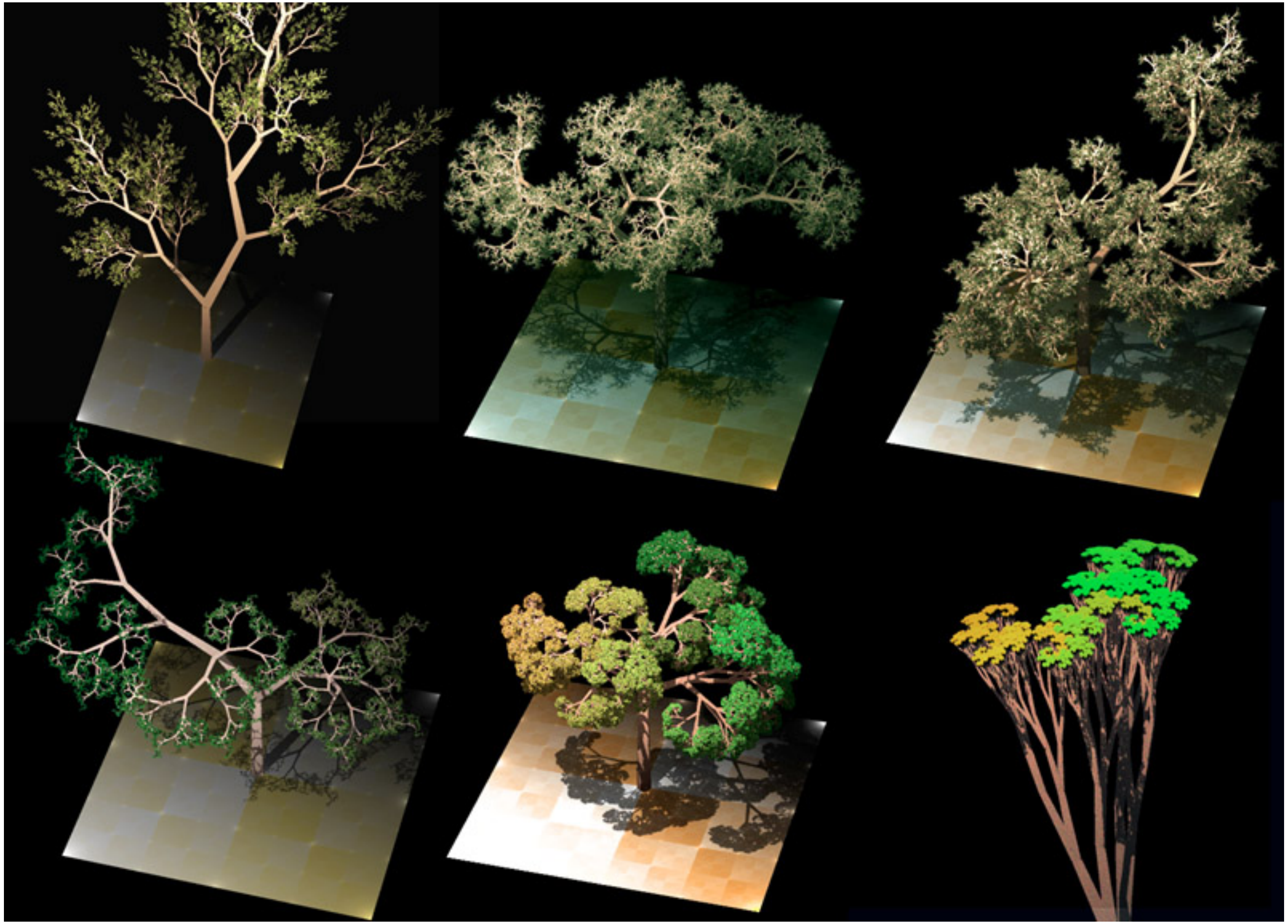
When t goes to infinity: $\dfrac{N_b(t)}{N_a(t)} = \dfrac{1+\sqrt{5}}{2} = \phi$

$$\begin{pmatrix} N_a(t) \\ N_b(t) \end{pmatrix} \simeq \begin{pmatrix} 0.276 \\ 0.447 \end{pmatrix} \left(\tfrac{1+\sqrt{5}}{2}\right)^t = (0.276 + 0.447) \begin{pmatrix} 0.382 \\ 0.618 \end{pmatrix} \left(\tfrac{1+\sqrt{5}}{2}\right)^t = 0.723 \begin{pmatrix} 0.382 \\ 0.618 \end{pmatrix} \left(\tfrac{1+\sqrt{5}}{2}\right)^t$$



$$\frac{0.618}{0.382} = \varphi$$

L-system trees form realistic models of natural patterns

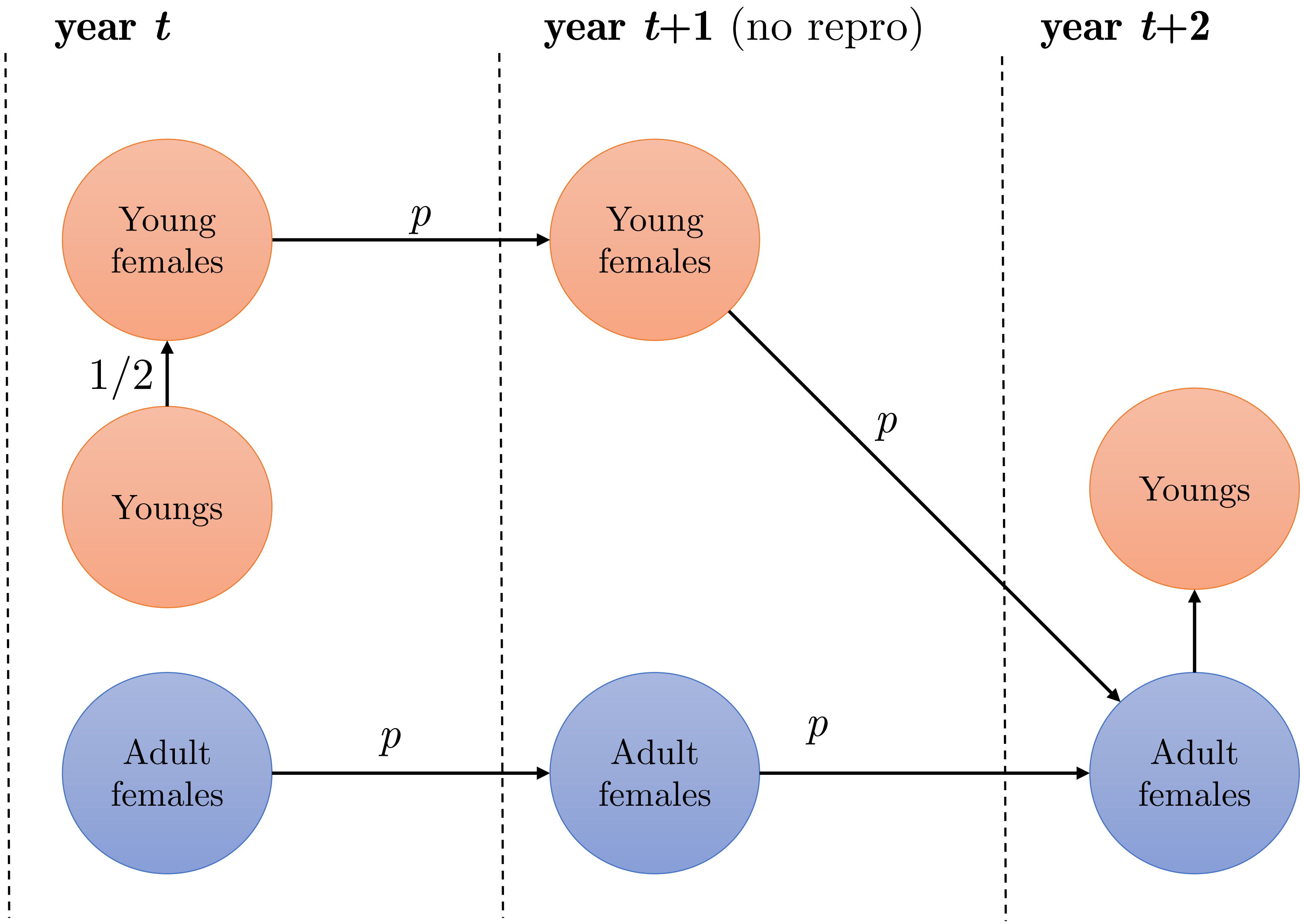


https://en.wikipedia.org/wiki/L-system

Ibex population dynamics from "Grand Paradis"

http://www.pngp.it/en

Ibex population dynamics from “Grand Paradis”



Ibex population dynamics from “Grand Paradis”

$$\mathbf{X}_t = \begin{pmatrix} J_t \\ A_t \end{pmatrix} \qquad \mathbf{X}_{t+2} = \begin{pmatrix} \frac{p^2}{2} & p^2 \\ \frac{p^2}{2} & p^2 \end{pmatrix} \mathbf{X}_t = p^2 \begin{pmatrix} 0.5 & 1 \\ 0.5 & 1 \end{pmatrix} \mathbf{X}_t$$

$$Nt = J_t + 2A_t$$

$$N_{t+2} = J_{t+2} + 2A_{t+2}$$

$$N_{t+2} = p^2\left((0.5J_t + A_t) + 2\left(0.5J_t + A_t\right)\right) = p^2\left(1.5J_t + 3A_t\right)$$

$$N_{t+2} = \tfrac{3}{2}p^2\left(J_t + 2A_t\right) = \tfrac{3}{2}p^2 N_t$$

$$\boxed{\frac{3}{2}p^2 > 1 \Leftrightarrow p > 0.82}$$

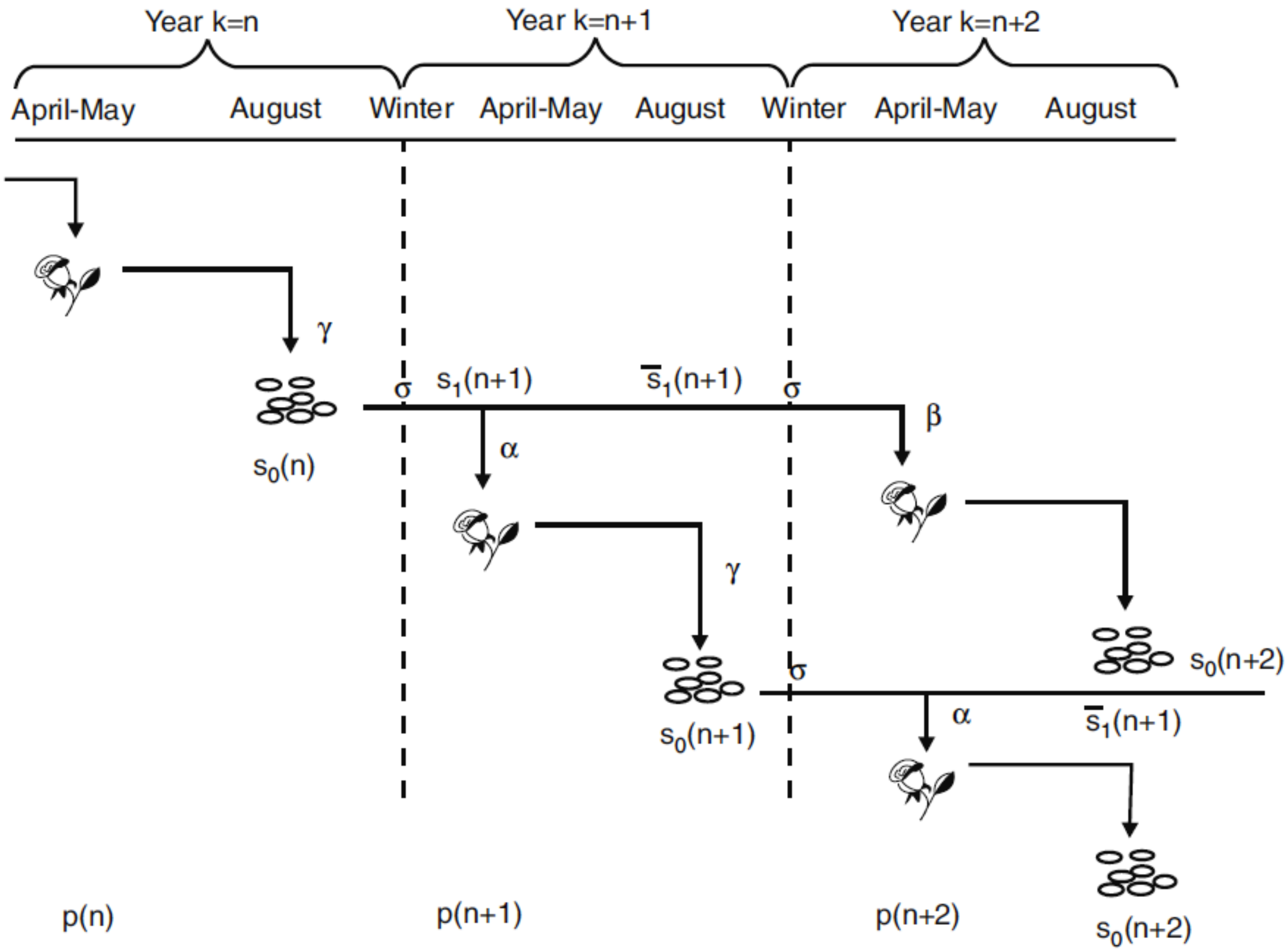
Propagation of annual plants

- Plants produce seeds at the end of their growth season (say August), after which they die.
- Only a fraction of these seeds survive the winter, and those that survive germinate at the beginning of the season (say May), giving rise to a new generation of plants.

Let

$$\begin{aligned}
\gamma &= \text{number of seeds produced per plant in August,}\\
\alpha &= \text{fraction of one-year-old seeds that germinate in May,}\\
\beta &= \text{fraction of two-year-old seeds that germinate in May,}\\
\sigma &= \text{fraction of seeds that survive a given winter.}
\end{aligned}$$

Propagation of annual plants (*continued*)



Propagation of annual plants (*continued*)

If $p(n)$ denotes the number of plants in generation n, then

$$p(n) = \begin{pmatrix} \text{plants from} \\ \text{one-year-old seeds} \end{pmatrix} + \begin{pmatrix} \text{plants from} \\ \text{two-year-old seeds} \end{pmatrix},$$

$$p(n) = \alpha s_1(n) + \beta s_2(n), \tag{2.7.1}$$

where $s_1(n)$ (respectively, $s_2(n)$) is the number of one-year-old (two-year-old) seeds in April (before germination). Observe that the number of seeds left after germination may be written as

$$\text{seeds left} = \begin{pmatrix} \text{fraction} \\ \text{not germinated} \end{pmatrix} \times \begin{pmatrix} \text{original number} \\ \text{of seeds in April} \end{pmatrix}.$$

This gives rise to two equations:

$$\tilde{s}_1(n) = (1 - \alpha)s_1(n), \tag{2.7.2}$$

$$\tilde{s}_2(n) = (1 - \beta)s_2(n), \tag{2.7.3}$$

Propagation of annual plants (*continued*)

where $\tilde{s}_1(n)$ (respectively, $\tilde{s}_2(n)$) is the number of one-year (two-year-old) seeds left in May after some have germinated. New seeds $s_0(n)$ (0-year-old) are produced in August (Figure 2.6) at the rate of γ per plant,

$$s_0(n) = \gamma p(n). \tag{2.7.4}$$

After winter, seeds $s_0(n)$ that were new in generation n will be one year old in the next generation $n+1$, and a fraction $\sigma s_0(n)$ of them will survive. Hence

$$s_1(n+1) = \sigma s_0(n),$$

or, by using formula (2.7.4), we have

$$s_1(n+1) = \sigma \gamma p(n). \tag{2.7.5}$$

Similarly,

$$s_2(n+1) = \sigma \tilde{s}_1(n),$$

which yields, by formula (2.7.2),

$$\begin{aligned} s_2(n+1) &= \sigma(1-\alpha)s_1(n), \\ s_2(n+1) &= \sigma^2\gamma(1-\alpha)p(n-1). \end{aligned} \tag{2.7.6}$$

Propagation of annual plants (*continued*)

Substituting for $s_1(n+1), s_2(n+1)$ in expressions (2.7.5) and (2.7.6) into formula (2.7.1) gives

$$p(n+1) = \alpha\gamma\sigma p(n) + \beta\gamma\sigma^2(1-\alpha)p(n-1),$$

or

$$p(n+2) = \alpha\gamma\sigma p(n+1) + \beta\gamma\sigma^2(1-\alpha)p(n). \qquad (2.7.7)$$

The characteristic equation (2.7.7) is given by

$$\lambda^2 - \alpha\gamma\sigma\lambda - \beta\gamma\sigma^2(1-\alpha) = 0$$

with characteristic roots

$$\lambda_1 = \frac{\alpha\gamma\sigma}{2}\left[1 + \sqrt{1 + \frac{4\beta}{\gamma\alpha^2}(1-\alpha)}\right],$$

$$\lambda_2 = \frac{\alpha\gamma\sigma}{2}\left[1 - \sqrt{1 + \frac{4\beta}{\gamma\alpha^2}(1-\alpha)}\right].$$

Propagation of annual plants (*continued*)

Observe that λ_1 and λ_2 are real roots, since $1 - \alpha > 0$. Furthermore, $\lambda_1 > 0$ and $\lambda_2 < 0$. To ensure propagation (i.e., $p(n)$ increases indefinitely as $n \to \infty$) we need to have $\lambda_1 > 1$. We are not going to do the same with λ_2, since it is negative and leads to undesired fluctuation (oscillation) in the size of the plant population. Hence

$$\frac{\alpha\gamma\sigma}{2}\left[1 + \sqrt{1 + \frac{4\beta}{\gamma\alpha^2}(1-\alpha)}\right] > 1,$$

or

$$\frac{\alpha\gamma\sigma}{2}\sqrt{1 + \frac{4\beta(1-\alpha)}{\gamma\alpha^2}} > 1 - \frac{\alpha\gamma\sigma}{2}.$$

Squaring both sides and simplifying yields

$$\gamma > \frac{1}{\alpha\sigma + \beta\sigma^2(1-\alpha)}. \tag{2.7.8}$$

Propagation of annual plants (*continued*)

If $\beta = 0$, that is, if no two-year-old seeds germinate in May, then condition (2.7.8) becomes

$$\gamma > \frac{1}{\alpha\sigma}. \tag{2.7.9}$$

Condition (2.7.9) says that plant propagation occurs if the product of the fraction of seeds produced per plant in August, the fraction of one-year-old seeds that germinate in May, and the fraction of seeds that survive a given winter exceeds 1.

$$\boxed{\alpha\sigma\gamma > 1}$$