INSA 3BIM - Biomathematics 3: advanced ODE

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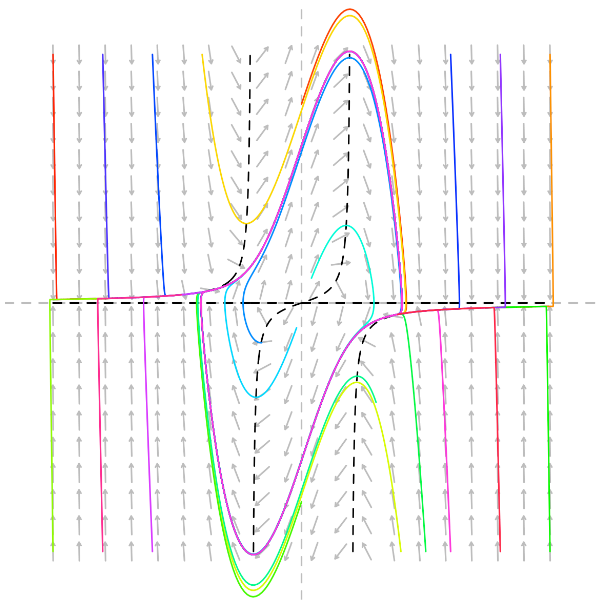

Van der Pol oscillator

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Models in $\mathbb{R}^2$

Almost a fish (edor2nonlin-intpre-02-EN)

Let Consider the following dynamical system:

$$\begin{cases} \frac{dx(t)}{dt} = 2y(t) \\ \frac{dy(t)}{dt} = 2x(t) - 3x(t)^2 \end{cases}$$

Perform the complete quailitative analysis (equilibrium points, stability, phase portrait). It mya be useful to introduce a function $H(x, y)$ to be determined in order to confirm the existence of centers around one of the equilibrium points.

The phase phase can be improved by specifically studying representative curves of $H(x, y) = 0$.

Lyapunov function (edor2nonlin-lyapunov-02-EN)

Study the following second order equation according to parameter b:

$$\frac{dx^2(t)}{dt^2} + b\left(\frac{dx(t)}{dt}\right)^3 + x(t) = 0$$

It may be useful to transform this equation into a system of first order by using $y(t) = \frac{dx(t)}{dt}$.

Prey-predator model (edor2nonlin-poinbend-06-EN)

We propose the following system of ODE to model the population dynamics of two species of size $N(t)$ et $P(t)$ (arbitrary units):

$$\begin{cases} \dfrac{dN(t)}{dt} = N(t)(2 - N(t)) - \dfrac{P(t)N(t)}{1 + N(t)} \\ \dfrac{dP(t)}{dt} = \dfrac{P(t)N(t)}{1 + N(t)} - \mu P(t) \end{cases}$$

where μ is a parameter such as $0 < \mu < \frac{2}{3}$.

Give a biological meaning of the model. Answer to the following questions in one or two short sentences.

1. What is the relation type between both species?
2. What does the expression $\frac{N(t)}{1+N(t)}$ represent in this model? How is this quantity evolve with $N(t)$?

Qualitative analysis of the model. Answer to the following questions by using the phase portrait given at the end, that you will fill in by using colors.

3. Give equations of the horizontal nullclines ($\frac{dP(t)}{dt} = 0$). Justify their representative curves.
4. Give equations of the vertical nullclines ($\frac{dN(t)}{dt} = 0$). Justify their representative curves.
5. Draw nullclines on the phase portrait below.

There are three equilibrium points denoted $\mathbf{A}_1$, $\mathbf{A}_2$ et $\mathbf{A}_3$. The coordinates of the first two equilibrium points (said "trivial") are:

- $\mathbf{A}_1$: $(N = 0, P = 0)$
- $\mathbf{A}_2$: $(N = 2, P = 0)$

6. What are the coordinates of the third equilibrium point said "non trivial" $\mathbf{A}_3$: (N^*, P^*).

7. Give the Jacobian matrix of the system.

8. Specify the nature of equilibrium point $\mathbf{A}_1$: $(N = 0, P = 0)$.

At the second equilibrium point $\mathbf{A}_2$: $(N = 2, P = 0)$, the Jacobian matrtrix writes:

$$\mathbf{J}_{\mathbf{A}_2} = \begin{pmatrix} -2 & -\frac{2}{3} \\ 0 & \frac{2}{3} - \mu \end{pmatrix}$$

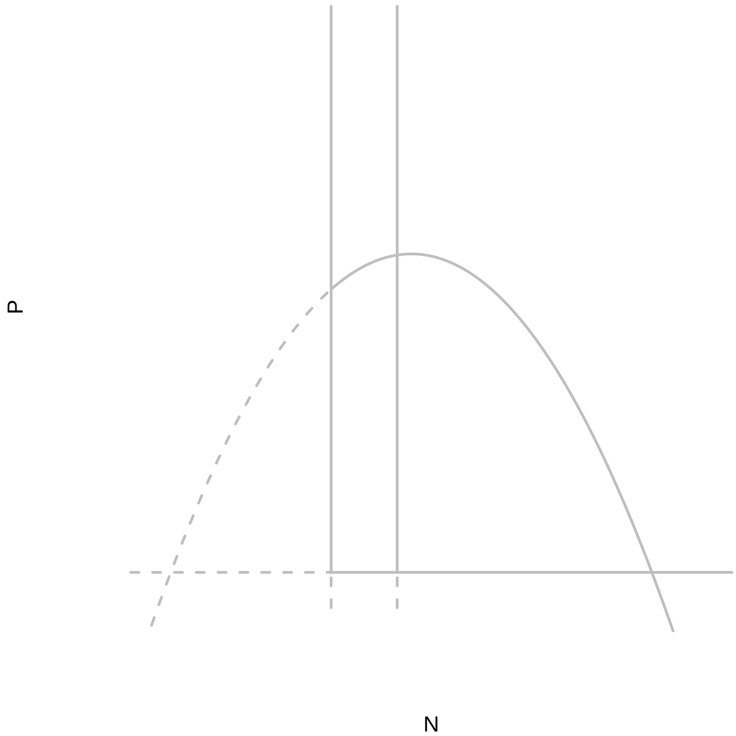
9. Specify the nature of equilibrium point $\mathbf{A}_2$: $(N = 2, P = 0)$.

At equilibrium point $\mathbf{A}_3$: (N^*, P^*), the Jacobian matrtrix writes:

$$\mathbf{J}_{(N^*,P^*)} = \begin{pmatrix} \frac{\mu(1-3\mu)}{1-\mu} & -\mu \\ 2 - 3\mu & 0 \end{pmatrix}$$

10. Conclude on the stability conditions of equilibrium point $\mathbf{A}_3$: (N^*, P^*) according to parameter μ. We will not specify the exact nature of $\mathbf{A}_3$ – spiral, node. . . – and we will not consider the characterisation of potential centers.

The case showed on the phase portrait below corresponds to $\mu = \frac{7}{24}$.



11. What is the stability of $\mathbf{A}_3$ when $\mu = \frac{7}{24}$? Fill in the phase portrait with the vector field and some appropriate trajectories.

12. Is it possible to highlight a limit cycle? If yes, specify the trapping region allowing to use the Poincaré Bendixon's theorem and justify your reasoning.

Plant-pollinator system (bifr2-02-EN)

Let consider the following plant-pollinator:

$$\begin{cases} \frac{da(t)}{dt} = a(t)(k - a(t)) + \frac{a(t)p(t)}{1+p(t)} & \text{(pollinators)} \\ \frac{dp(t)}{dt} = -\frac{p(t)}{2} + \frac{a(t)p(t)}{1+p(t)} & \text{(plants)} \end{cases}$$

with $k > 0$.

a) Give a biological meaning of the equations.

b) Give the coordinates of the equilibrium points and study their local stability.

c) Draw the phase portrait when $k > \frac{1}{2}$ or $-1 + \sqrt{2} < k < \frac{1}{2}$.

(bifhopf-02-EN)

Let consider the following equation:

$$\frac{d^2x(t)}{dt^2} + (x(t)^2 - \mu)\frac{dx(t)}{dt} + 2x(t) + x(t)^3 = 0$$

1) Transform the previous equation in a system of two first order differential equations with variables $(x(t), y(t))$, where $y(t) = \frac{dx(t)}{dt}$.

2) Give the coordinates of the equilibrium point(s).

3) Give the linear part of the system.

   Perform the study of the local stability in the neighbourhood of the equilibrium point(s).

   Give their nature.

   Draw the different possible cases along a μ-axis.

   Demonstrate that there exists a value of μ (denoted μ^*) corresponding to a change in the lass of topological equivalence of one of the equilibrium points.

4) Are the conditions of the Poincarré-Andronov-Hopf's theorem gathered together?

5) Study the local stability of the equilibrium point(s) when $\mu = \mu^*$.

6) Conclude.

7) Draw the phase portrait for $\mu = +1$.

Remember that the Mardsen-McCracken's index writes:

$$I = \omega^*[f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}] + [-f_{xy}(f_{xx} + f_{yy}) + g_{xy}(g_{xx} + g_{yy}) + f_{xx}g_{xx} - f_{yy}g_{yy}]$$

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