Biomathematics 1
ODE modelling in Life Sciences
The inter-specific competition model

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The inter-specific competition model

Hypotheses of the model

Here we consider two competing species for which we will build a dynamical model to better understand possible outputs linked to their interaction.

The basic hypotheses are :

- A logistic-type growth for each species when there is no competition, with parameters r_i and K_i, $i = 1, 2$;
- Each individual of species 1 inhibits the growth of individuals of species 2 in proportion α of individuals from species 2.
- Each individual of species 2 inhibits the growth of individuals of species 1 in proportion β individuals from species 1.

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Model equations

Species 1

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1 + \alpha N_2}{K_1}\right)$$

Species 2

$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{N_2 + \beta N_1}{K_2}\right)$$

Growth is of logistic-type for species 1 and 2 when there is no interaction, that is when $\alpha = 0$ and $\beta = 0$.

We will perform the qualitative analysis of this model in phase plane (N_1, N_2).

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The phase plane

- In phase plane (N_1, N_2), the flow field is defined by vectors of coordinates $(\frac{dN_1}{dt}, \frac{dN_2}{dt})$.
- The sign of $\frac{dN_1}{dt}$ will give information on the x-coordinate of each vector.
- The sign of $\frac{dN_2}{dt}$ will give information on the y-coordinate of each vector.
- Hence, because $\frac{dN_1}{dt}$ and $\frac{dN_2}{dt}$ are defined by the ODE system itself, this study is made possible.
- As a reminder, trajectories (namely, solution curves) follow the flow pattern of the direction field (tangency).
- Because we are dealing with autonomous equations, trajectories do never intersect.

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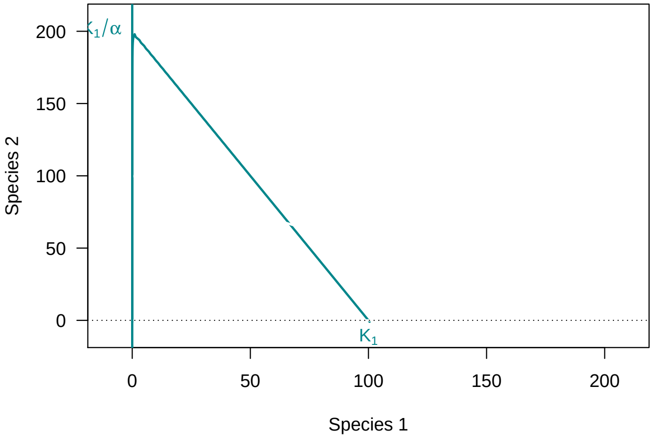
Vertical nullclines

As a reminder, vertical nullclines are curves in the phase plane along which vectors of the flow field are strictly vertical, meaning that vertical nullclines are solution curves of equation $\frac{dN_1}{dt} = 0$.

$$\begin{array}{ll} & \frac{dN_1}{dt} = 0 \\ \Leftrightarrow & r_1 N_1 \left(1 - \frac{N_1 + \alpha N_2}{K_1}\right) = 0 \\ \Leftrightarrow & N_1 = 0 \\ \text{or} & \frac{N_1 + \alpha N_2}{K_1} = 1 \\ \Leftrightarrow & N_1 + \alpha N_2 = K_1 \\ \Leftrightarrow & N_2 = \frac{K_1 - N_1}{\alpha} \end{array}$$

So, there are two vertical nullclines:

- $N_1 = 0$: the y-axis (*i.e.*, the vertical axis);
- $N_2 = \frac{K_1 - N_1}{\alpha}$: a decreasing straight line of slope $-\frac{1}{\alpha}$ going through the point of coordinates $(K_1, 0)$. The intercept with the y-axis is located at $\frac{K_1}{\alpha}$.


K1/α
K1
200
150
100
50
0
Species 2
0
50
100
150
200
Species 1

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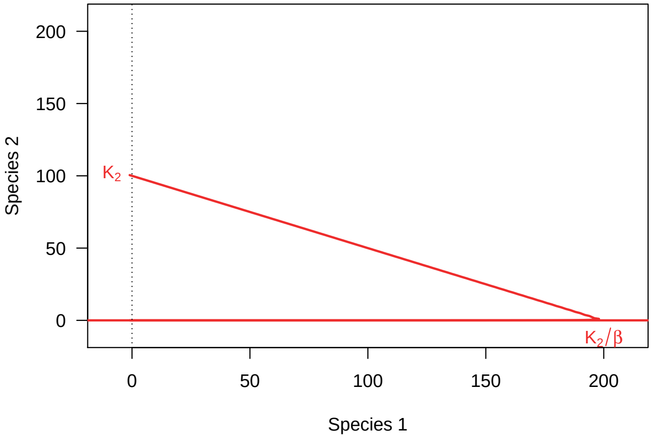
Horizontal nullclines

We can think in terms of symmetry of equations:
$N_1 \leftrightarrow N_2,\ K_1 \leftrightarrow K_2,\ r_1 \leftrightarrow r_2$ et $\alpha \leftrightarrow \beta$
Hence:

$$\begin{array}{rr} & \frac{dN_2}{dt} = 0 \\ \Leftrightarrow & N_2 = 0 \\ \text{ou} & N_1 = \frac{K_2 - N_2}{\beta} \\ \Leftrightarrow & N_2 = K_2 - \beta N_1 \end{array}$$

So, there are two horizontal nullclines:

- $N_2 = 0$: the x-axis (*i.e.*, the horizontal axis);
- $N_2 = K_2 - \beta N_1$: a decreasing straight line of slope $-\beta$ going through the point of coordinates $(0, K_2)$. The intercept with the x-axis is located at $\frac{K_2}{\beta}$.



200
150
100
50
0
Species 2
K2
K2/β
0
50
100
150
200
Species 1

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Combining nullclines

Plotting both vertical and horizontal nullclines on the same graph requires to consider different cases regarding the respective positions of K_1 and $\frac{K_2}{\beta}$, as well as K_2 and $\frac{K_1}{\alpha}$.

→ Give these possible cases.

→ Plot nullclines in each of these cases. Add directions of the flow field.

→ Under the hypothesis of this model, can both species possibly coexist?

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Search for equilibrium points

As a reminder, equilibrium points are constant solutions.

→ How many equilibrium points do you find for this model of inter-specific competition?

→ What are the conditions on parameters to ensure their biological meaning, that is to ensure that both N_1^* and N_2^* are positive?

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Stability of equilibrium points

→ Give the Jacobian matrix of the model?

→ For each equilibrium point, calculate the Jacobian matrix and determine its local stability. When possible, give also its nature (node, spiral,...).

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Phase portraits

→ For each of the different cases identified earlier, plot the phase portrait.

→ Add some trajectories

→ Give a biological interpretation of the results.

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Chronicles

→ In the particular case where both species can coexist, plot both chronicles for an initial condition of your choice.

→ Discuss the outputs on a biological point of view.