ODE modelling in Life Sciences

INSA - 3BS 2020/2021

Practical session 1

Introduction

All practical session regarding introduction to the use of Ordinary Differential Equation (ODE) to model life science phenomena will be performed under the R software.

R packages for ODE

You will be introduced to numerically solve first order ODEs with R. For that purpose, you will use package `phaseR` performing a qualitative analysis of one- and two-dimensional autonomous ordinary differential equation systems, using phase plane methods.

This packages is not installed by default in the basic R configuration. You need to install it on your computer:

```
install.packages("phaseR")
```

After installing the package, you need to load it within your current R session.

```
library("phaseR")
```

Please note that another package, named `deSolve`, may be useful to go in deeper use of numerical schemes to simulate ODE.

Population dynamics models

We will illustrate how to use R on the same models as the ones you analytically studied :

$$\frac{dN(t)}{dt} = N(t)g(N(t))$$

where $N(t)$ is the population density and $g(N(t))$ the intrinsic population growth rate.

Different models have been proposed :

1. Gompertz model with $g(N(t)) = r\ln(K/N(t))$
2. Allee effect model with $g(N(t)) = r(K - N(t))(N(t) - \alpha)$
3. Allee effect with exploitation, with $g(N(t)) = r(K - N(t))(N(t) - \alpha) - E$, with $\alpha < K$

where $r = 0.02$, K, α and E strictly positive.

Graphical representation of $g(N(t))$

The graphical representation of $g(N(t))$ for the Gompertz model is obtained with the following code:

```r
r <- 0.02
K <- ...
curve(r * log(K / x), from = 0, to = 150,
      xlab = "N(t)", ylab = "g(N(t))",
      col = "red", lwd = 2, las = 1)
abline(h = 0, lty = 3)
abline(v = 0, lty = 3)
```

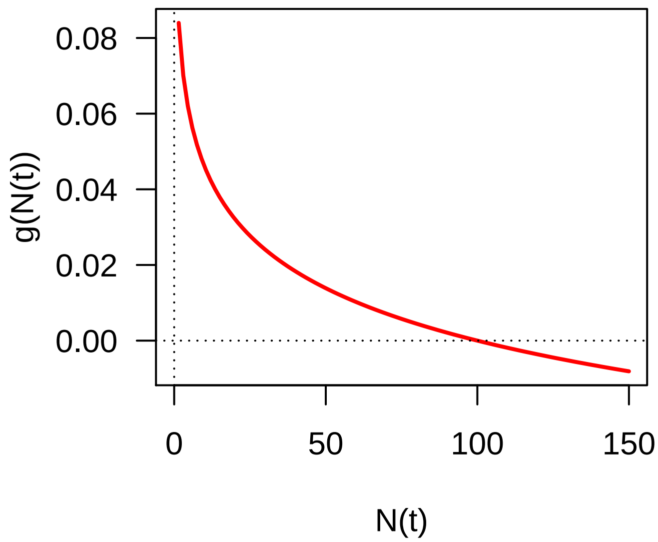
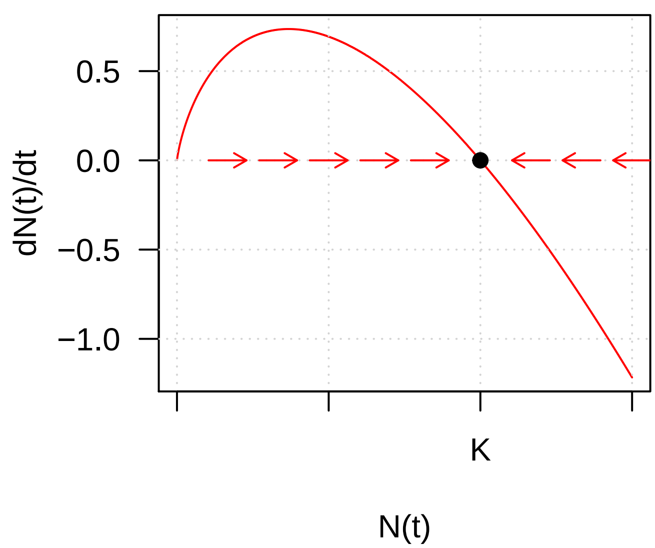


Figure 1: Graphical representation of the Gompertz population growth rate $g(N(t)) = r \ln K/N(t)$ as a function of $N(t)$.

1. From the above representation (and the qualitative analysis you did), deduce the value of parameter K that has been used. Complete and run the code to obtain Figure 1.

The phase plane of the Gompertz model is as follows:



2. How can you deduce the direction of red arrows from Figure 1? What is the stability of equilibrium point K?

The representation of $g(N(t))$ for the Allee effect model is given in Figure 2 for two values of parameter α:

```r
r <- 0.02
K <- 100
alpha1 <- ...
par(mfrow = c(1, 2)) # create a two-column plot
curve(r * (K - x) * (x - alpha1), from = 0, to = 150,
      ylim = c(-1, 20), xlab = "N(t)", ylab = "g(N(t))",
      col = "red", las = 1,
      main = expression(paste(alpha, "=...", sep = "")),
      cex.lab = 1.5, cex.main = 2, lwd = 2)
abline(h = 0,lty = 3)
abline(v= 0, lty = 3)

alpha2 <- ...
curve(r * (K - x) * (x - alpha2), from = 0, to = 200,
      ylim = c(-1, 20), xlab = "N(t)", ylab = "g(N(t))",
      col = "red", las = 1,
      main = expression(paste(alpha, "=...", sep = "")),
      cex.lab = 1.5, cex.main = 2, lwd = 2)
abline(h = 0,lty = 3)
abline(v= 0, lty = 3)
```

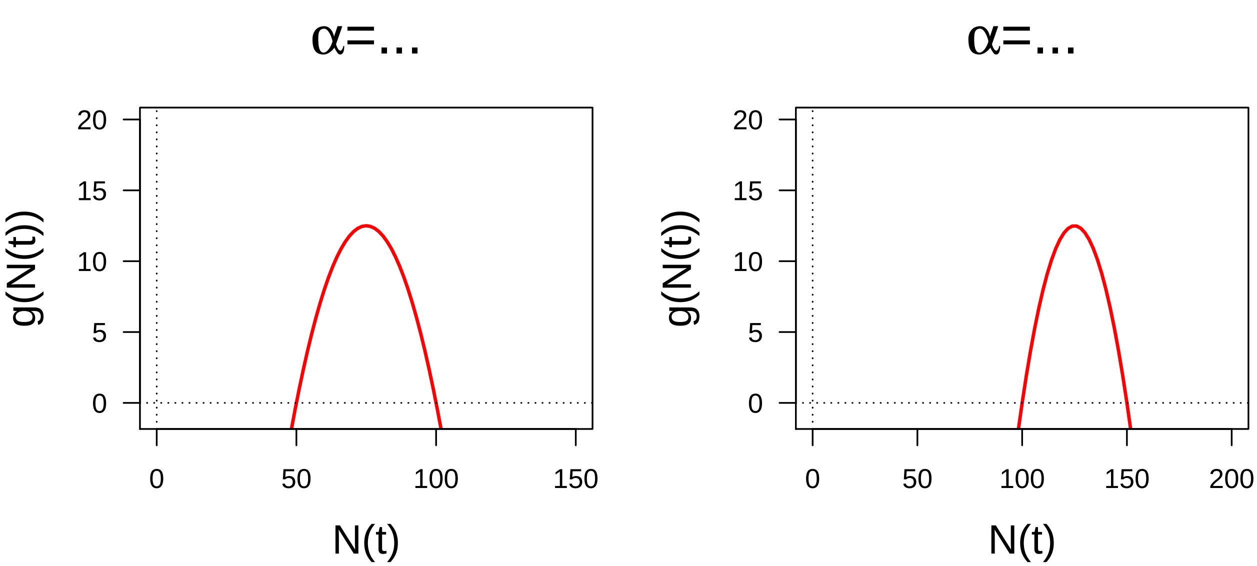


Figure 2: Graphical representation of the Allee effect model with $g(N(t)) = r(K - N(t))(N(t) - \alpha)$ as a function of $N(t)$ for two values of parameter α.

3. Deduce from Figure 2 (and the qualitative analysis you did), which value of α has been used in each case. Draw by hand the corresponding phase portraits and deduce the stability of the equilibrium points in each case.

4. Get inspired from the previous codes, draw function $g(N(t))$ as a function of $N(t)$ for the Allee effect model with exploitation, taking the following parameter values: $r = 0.02$, $K = 100$, $\alpha = 50$ and $E = 0.05$.

5. Change value of parameter E to 15 and add the corresponding curve of $g(N(t))$ on the previous plot; for that purpose, you can use option `add = TRUE` in `curve()`function. What do you notice regarding equilibrium points?

Numerical simulations

In this last part, we will simulate the time course of $N(t)$ with the R software, in particular with the `phaseR` package. For that purpose, we need to perform the following steps, illustrated with the Allee effect model with exploitation.

- Define a function implementing the model as an ODE. The general format of the derivative function (ODE) should be a common one: given `t`, `y` and a vector of parameters, it should simply return a `list` with the first element being the value of the derivative (`dy`).
- Assign values to parameters
- Define an initial condition, i.e. an initial value of N at $t = 0$: $N(t = 0) = N_0$.
- Numerically solve the ODE to get predictions.
- Plot the flow field
- Add horizontal lines at equilibrium points
- Add trajectories (namely, predictions)

```
library(phaseR)
# Define the model as an ODE
Aexpmod <- function(t, y, parameters){ # Allee effect model
  r <- parameters[1] # Intrinsic population growth rate
  K <- parameters[2] # Carrying capacity
  alpha <- parameters[3] # Allee threshold
  E <- parameters[4] # Exploitation parameter
  dy <- (r * (K - y) * (y - alpha) - E) * y
  list(dy)
}
# Assign values to parameters
r <- 0.02
K <-  10
alpha <- 5
E <- 0.001
parameters <- c(r, K, alpha, E)
# Choose an initial condition
Ninit <- c(30)
# Plot the flow field
tmax <- 10
Aexpff <- flowField(deriv = Aexpmod, xlim = c(0, tmax), ylim = c(0, 31),
          parameters = parameters, system = "one.dim",
          state.names = "N(t)",
          add = FALSE) # Create a new plot
# Add nullclines
Aexpnc <- nullclines(deriv = Aexpmod, xlim = c(0, tmax), ylim = c(0, 31),
           parameters = parameters, system = "one.dim",
           add = TRUE, add.legend = FALSE)
# Add trajectories
Aexpt <- trajectory(deriv = Aexpmod, y0 = Ninit, tlim = c(0, tmax),
           parameters = parameters, system = "one.dim",
           add = TRUE, col = "red")
```

6. Run all commands as given above. From the top right panel `Environment`, explore object `Aexpt`: what type of object is it? What sub-objects does it contain? What is `Aexpt$y`? What about its dimension?

7. Model predictions are given in Figure 3 below. Reproduce this figure and biologically interpret the output.

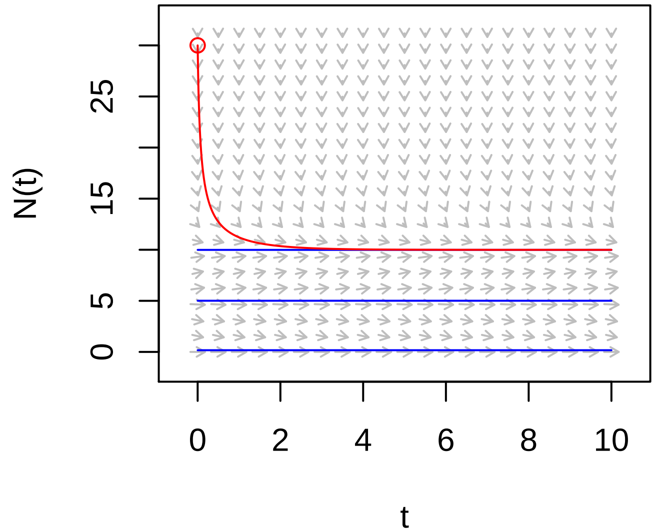


Figure 3: predictions of the Allee effect model with exploitation for $r = 0.02$, $K = 10$, $\alpha = 5$, $E = 0.001$ and $N(0) = 30$.

8. What happens for $N(0) = 5$? Add the corresponding model prediction to the previous graph. If necessary, extend the time window until 20.

9. Redo the previous numerical simulations for $E = 0.2$. What do you observe? Interpret the outputs.