Advanced ODE modelling in Life Sciences

INSA - 3BIM - Biomathematics 3

Practical session with R: Bazikin's model

Introduction

In this practical exercise, we model the time course of a plant biomass consumed by herbivores with the following system of ODE:

$$\begin{cases} \frac{dx(t)}{dt} = x(t) - \frac{x(t)y(t)}{1+\alpha x(t)} - \varepsilon x(t)^2 \\ \\ \frac{dy(t)}{dt} = -\gamma y(t) \left(1 - \frac{x(t)}{1+\alpha x(t)}\right) \end{cases} \tag{1}$$

We assume that parameter $\varepsilon \geq 0$, and for simplicity reasons, we fix $\alpha = 0.5$ and $\gamma = 1$.

The objective is therefore to study the dynamics of both populations according to parameter ε.

1. Give a biological interpretation of both equations as initially written with the three parameters.
2. Find the equilibrium points and specify under which conditions on parameters they exist.
3. Write an `R` function taking parameter ε as argument and returning a `dataframe` with each row corresponding to the equilibrium points of system (1) according to the input ε value.

Cas (I): $\varepsilon = 0$

4. Calculate the Jacobian matrix of system (1) and specify both the stability and the nature of all equilibrium points.
5. Using function `stability` within library `phaseR`, check your previous calculations of Jacobian matrices for each equilibrium point.
6. Plot the phase portrait within the phase plane (x, y) and both chronicles for the following initial conditions:
   - $x(0) = 1$ and $y(0) = 10$
   - $x(0) = 20$ and $y(0) = 5$
7. Give your conclusions.

Cas (II): $\varepsilon \neq 0$

8. Calculate the Jacobian matrix of system (1) and specify both the stability and the nature of all equilibrium points.
9. For the non-trivial equilibrium point $(x^*, y^*) \neq (0, 0)$, detail the different possibilities according to ε values and display them along an ε axis. Justify why we can suspect a Poincaré-Andronov-Hopf bifurcation, and specify for with bifurcation value ε^*.

10. Check if the application conditions of the Poincaré-Andronov-Hopf theorem are verified.
11. If you answered `yes` to the previous question, calculate the Marsden-MacCracken index[1] to study the stability of the non-trivial equilibrium point at bifurcation value ε^*. We will assume here that the calculation of this index remains valid without bringing the equilibrium point back to the origin.
12. What type of bifurcation is this?
13. Using function `stability` within library `phaseR`, check your stability conclusion for all equilibrium points with the following ε values:
    - $\varepsilon = 0.1$
    - $\varepsilon = 1/6$
    - $\varepsilon = 0.25$
    - $\varepsilon = 0.4$
    - $\varepsilon = 1$
14. How does `R` proceed to return the stability of the non-trivial equilibrium point for $\varepsilon = 1/6$?
15. For all the above ε values, plot both phase portraits and chronicles with the initial condition $x(0) = 10$ and $y(0) = 3$. Provide in each case a biologically meaningful conclusion.

[1] We remind that the Marsden-MacCracken index writes as follows:

$$I = \omega^* \left(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}\right) - f_{xy}(f_{xx} + f_{yy}) + g_{xy}(g_{xx} + g_{yy}) + f_{xx}g_{xx} - f_{yy}g_{yy}$$