BioInformatics and Modelling (4BIM - M1)
Game theory

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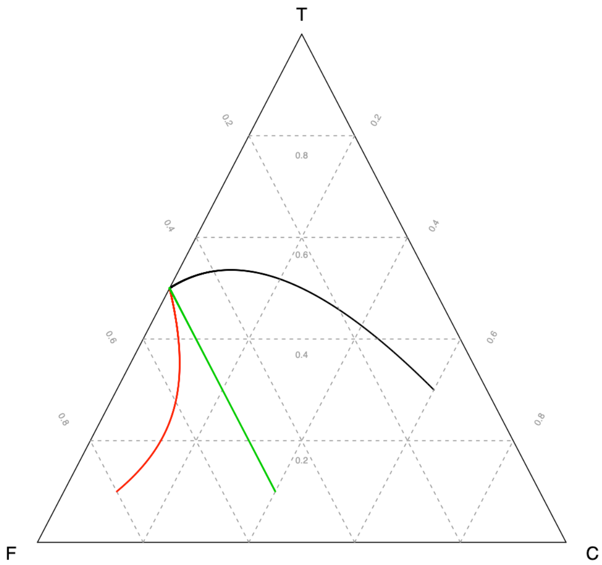

Une variante du jeu Faucon-Colombe

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1 Game theory

1.1 Hawkes, Doves, Bourgeois (theojeux-01)

Let consider a new strategy B (bourgeois) in addition to the classical ones: Hawkes (F) and doves (C). Bourgeois use strategies F and C in equal proportions $(\frac{1}{2}, \frac{1}{2})$ with the following gain when two bourgeois meet together:

$$E(B,B) = \frac{1}{2}E(F,C) + \frac{1}{2}E(C,F) \tag{1}$$

with E(i,j) the gain of i against j.

1) Write the gain matrix (3x3) of the game (F,C,B).

2) Give the equations fo the replicator. We denote x, y and z the relative proportions of F, C and B.

3) Rewrite the previous system in $\mathbb{R}^2$.

4) Find equilibrium points and study their local stability.

5) Draw the phase portrait in the triangle defined by $x + y + z = 1$. Direct with justification the vectors on the triangle sides.

6) Is strategy B an ESS?

1.2 Exam March 2004 (theojeux-02)

Let consider the following gain matrix:

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & 3 \\ -1 & 0 & 3 \\ 1 & 1 & 0 \end{pmatrix} \tag{2}$$

1) Write the replicator equations for variables $x(t), y(t), z(t)$.

2) Rewrite the previous system in dimension 2.

3) Find equilibrium points.

4) Study the local stability of the equilibrium points.

5) Draw the phase portrait in the triangle $x+y+z = 1$. Add some trajectories and potential separatrices.

1.3 Exam June 2002 (theojeux-03)

From Hofbauer and Sigmund (2003)[1].
Let consider the following gain matrix:

$$A = \begin{pmatrix} 0 & 6 & -4 \\ -3 & 0 & 5 \\ -1 & 3 & 0 \end{pmatrix} \tag{3}$$

[1] Hofbauer J, Sigmund K. 2003. Evolutionary game dynamics. *Bulletin of the American Mathematical Society*, 40:479–519.

1) The game corresponding to the matrix above implies three strategies 1, 2 and 3. Calculate the mean gain within a population comprised of proportions $x(t)$, $y(t)$ and $z(t)$ that play strategies 1, 2 and 3, respectively ($x + y + z = 1$). Recall that the mean gain is given by:

$$\Delta = (x, y, z)A \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{4}$$

2) Calculate the gain of an individual playing always the same strategy 1, 2 or 3 against the population of proportions $x(t)$, $y(t)$ and $z(t)$. These gains will be denoted $\Delta_1, \Delta_2, \Delta_3$, respectively:

$$\Delta_1 = (1, 0, 0)A \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \Delta_2 = (0, 1, 0)A \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \Delta_3 = (0, 0, 1)A \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{5}$$

3) Calculate the mean gain of the population, which is given by:

$$\Delta = x\Delta_1 + y\Delta_2 + z\Delta_3 \tag{6}$$

4) Write the replicator equations in dimension 3.

5) Write the previous equations in dimension 2 by using $z = 1 - x - y$.

6) Find all equilibrium points, at the vertices and along the sides of the unit triangle.

   For both sides of the unit triangle corresponding to $z = 0$ and $y = 0$, write the equation which governs variable $x(t)$ in the following form: $\frac{dx}{dt} = f(x)$. In each case, determine function f.

   Find the equilibrium points for these equations and determine their stability.

   In the same way, for side $x = 0$, write the equation $\frac{dy}{dt} = g(y)$. Answer to the same questions.

   Find also an interior non trivial equilibrium point.

7) Study the stability of all equilibrium points.

8) Draw the phase portrait in the unit triangle.

9) What is the asymptotic behaviour of the system?

1.4 Exam April 2001 (theojeux-04)

Let consider the following gain matrix:

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \tag{7}$$

1) Write the replicator equations for variables $x(t)$, $y(t)$ and $(z(t)$.

2) Show that the previous system writes in the following way in $\mathbb{R}^2$ ($x + y + z = 1$):

$$\begin{cases} \frac{dx}{dt} = -x[1 - x + 2(y - x)(1 - x - y)] \\ \frac{dy}{dt} = y[1 - y - 2(y - x)(1 - x - y)] \end{cases} \tag{8}$$

3) Find the equilibrium points.

4) Study the local stability in the neighborhood of equilibrium points.

5) Draw the phase portrait in the triangle $x+y+z=1$. Add some trajectories and potential separatrices.

1.5 Exam Decembre 1997 (theojeux-05)

Let consider the following gain matrix:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{9}$$

1) Write the replicator equations for variables $x(t)$, $y(t)$ and $z(t)$ ($x+y+z=1$). It can be convenient to use function $\sigma = xy + xz + yz$.

2) Rewrite the previous system in $\mathbb{R}^2$.

3) Find the equilibrium points.

4) Study the local stability in the neighborhood of the equilibrium points.

5) We want now to demonstrate that the equilibrium point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is globally asymptotically stable.

   a) Show that function $V(x,y,z) = xyz$ is a Lyapunov's function for the system and the equilibrium point under study.

   b) Show that $\dot{V}$ has a constant sign within the unit triangle.

   c) Conclude.

6) Draw the phase portrait in the triangle $x+y+z=1$.

1.6 Exam Spring 2016 (theojeux-08)

Exercise 1

In *Theoretical Population Biology*, Apaloo (2006) describes a game with three strategies (A, B et C) and defines the associated gain matrix[2]:

$$G = \begin{pmatrix} 3 & 2 & 5 \\ 5 & 1 & 3 \\ 4 & 4 & 1 \end{pmatrix}$$

Let denote x, y et z the proportions of A, B and C, respectively.

1. Is strategy B an ESS? Justify.
2. Calculate the mean gains of strategies A, B and C, respectively denoted Δ_A, Δ_B and Δ_C.
3. Calculate the mean gain within the population, denoted Δ.

[2] Apaloo J. 2006. Revisiting matrix games: The concept of neighborhood invader strategies. *Theoretical Population Biology*, 69:235–242.

4. Write the replicator equations in dimension 3.

   In dimension 2, the system writes as follows:

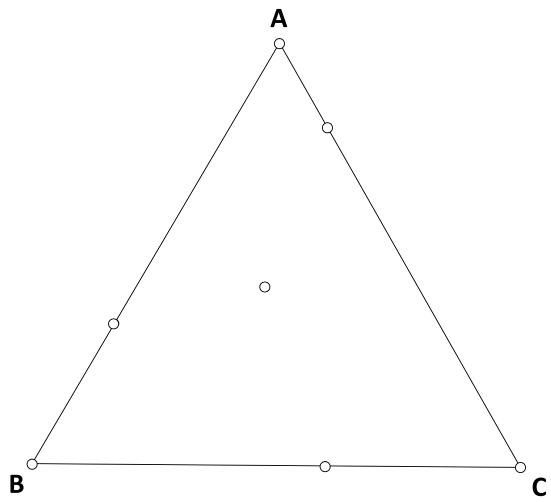
$$\begin{cases} \dot{x} = x(5x^2 + 5y^2 - 9x - 8y + 7xy + 4) \\ \dot{y} = y(5x^2 + 5y^2 - 5x - 7y + 7xy + 2) \end{cases} \tag{10}$$

5. Find equilibrium points and give their coordinates in plane $x + y + +z = 1$. We will denote them $P_1 = (x^*, 0, 0)$, $P_2 = (0, y^*, 0)$, $P_3 = (0, 0, z^*)$, $P_4 = (0, y^*, z^*)$, $P_5 = (x^*, 0, z^*)$, $P_6 = (x^*, y^*, 0)$ and $P_7 = (x^*, y^*, z^*)$, respectively. Show that $P_6 = (1/3, 2/3, 0)$ and $P_7 = (8/19, 6/19, 5/19)$.
6. Give the Jacobian matrix of the system in dimension 2.
7. Study the nature and the stability of equilibrium points P_1, P_2, P_3, P_4, P_5 and P_6.

   The Jacobian matrix at equilibrum point P_7 is:

$$J_7^* = -\frac{2}{361}\begin{pmatrix} 196 & 144 \\ -81 & 51 \end{pmatrix}.$$

8. Determine the nature and the stability of equilibrium point P_7.
9. Complete the phase portrait below, in the unit triangle (circles correspond to equilibrum points). Justify the direction of vectors along the triangle sides. Draw some trajectories and discuss the results.



Exercise 2

We are here interested in the conflict between two endosymbiotic bacterial species of a drosophila population, regarding nutriment sharing within their host. These two bacterial species, called A et B, represent two strains each: A_1/A_2 and B_1/B_2. Both strains of a same species do not interact the one with the other, but interact with the strains of the other species:

- The presence of a bacterium B_1 does not influence the fitness of a bacterium A_1 but increases the fitness of a bacterium A_2 by a factor 3;
- The presence of a bacterium B_2 increases the fitness of a bacterium A_1 by a factor 4 while it increases the fitness of a bacterium A_2 by a factor 1;
- The presence of a bacterium A_1 increases the fitness of a bacterium B_1 by a factor 2 while it increases the fitness of a bacterium B_2 by a factor 1;
- The presence of a bacterium A_2 increases the fitness of a bacterium B_1 by a factor 4 while it does not influence the fitness of a bacterium B_2.

Let A be the gain matrix of bacteria of species A against bacteria of species B, et B the one of bacteria of species B against bacteria of species A. Let denote x the proportion of bacteria of strain A_1 (so that $(1-x)$ is the proportion of bacteria of strain A_2), and y the proportion of bacteria of strain B_1 (so that $(1-y)$ is the proportion of bacteria of strain B_2).

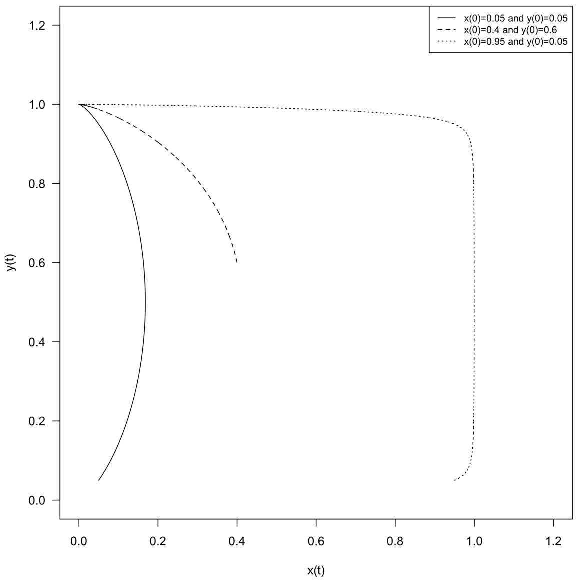
1. Fill in the following gain matrices:

   $A = \begin{pmatrix} .... & 4 \\ 3 & .... \end{pmatrix}$ $\qquad$ $B = \begin{pmatrix} .... & 4 \\ 1 & .... \end{pmatrix}$

2. Gives the mean gains of the different strategies, denoted Δ_{A_1}, Δ_{A_2}, Δ_{B_1} and Δ_{B_2}, respectively.

3. Calculate the mean gain within the population of species A, denoted Δ_A, and within the population of species B, denoted Δ_B.

   The replicator equations of this model allows to follow the dynamics of variables $x(t)$ and $y(t)$. In the phase plane (x, y), the system has four equilibrium points: $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$. The phase portrait of the model is given below for different initial conditions.



4. Give a biological interpretation of the results regarding the outcome of the competition between he two bacterial species.

1.7 Exam Spring 2017 (theojeux-09)

Exercise 1
Let consider a new strategy O (Opponents) in addition to the classical strategies Hawkes (F) and Doves (C). Opponents systematically adopt the opposite strategy to the one of their opponent. Let denote $a(i,j)$ the gain of strategy i against strategy j. Let make the following hypothesis:

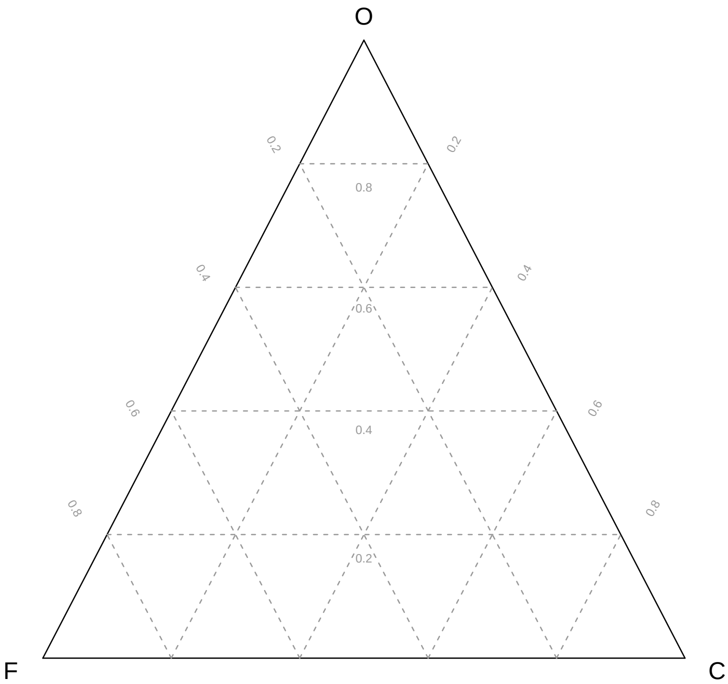
$$a(O,O) = \frac{1}{2}a(F,C) + \frac{1}{2}a(C,F)$$

Here we consider that the cost linked to time loss in fighting by hawkes (T) is equal to 0.
Let denote x, y and z the relative proportions of each strategy with $x + y + z = 1$.

1. Write matrix $\mathbf{A}$ of gains for the three strategies. Justify.
2. Calculate the individual gains of strategies F, C and O. Please denote these gains Δ_F, Δ_C and Δ_O, respectively.
3. Check that the mean gain equals $\Delta = \frac{G}{2} - \frac{Cx^2}{2}$.
4. Write the replicator equations in dimension 3.
5. Demonstrate that the replicator equations in dimension 2 are the following:

$$\begin{cases} \dot{x} = x\left(\frac{C}{2}x^2 - \frac{G+C}{2}x + \frac{G}{2}\right) \\ \dot{y} = y\left(\frac{C}{2}x^2 + \frac{G}{2}y - \frac{G}{2}\right) \end{cases} \tag{11}$$

   What do you notice?
6. For which value(s) of x, does the first equation (the $\dot{x}$ one) become zero?
7. Deduce the equilibrum points from the previous question. Show that there are five equilibrium points according to some conditions on parameters G and C to be specified.
8. Calculate the Jacobian matrix of the system.
9. Under condition $G < C$, determine the local stability of the equilibrium points.
10. Let fix $G = 1.8$ and $C = 3$. Complete the phase portrait below by ploting the five equilibrium points. Mathematically justify the direction of vectors on the triangle sides. Plot the potential separatrices and draw some trajectories for the following initial conditions:
    - $x(0) = 0.8$, $y(0) = 0.1$ and $z(0) = 0.1$ in black;
    - $x(0) = 0.1$, $y(0) = 0.8$ and $z(0) = 0.1$ in green;
    - $x(0) = 0.1$, $y(0) = 0.3$ and $z(0) = 0.6$ in red.



11. What can you deduce about the system dynamics?

Exercise 2

Two drivers A and B point their cars towards each other on a street road that is too narrow for them to pass each other without causing an accident. If a driver slows down (strategy R), while the other keeps the same speed (strategy N), he loses face: he has then a gain a, while his opponent has a gain b. If both drivers slow down together, then both get the same gain c. At last, if none of the drivers slows down, then an accident occurs and each driver has a gain d.

Let denote x the proportion of drivers in the population with strategy R, and y the proportion of drivers with strategy N ($x + y = 1$).

12. Give the gain matrix associated to this game according to parameters a, b, c and d.

13. Give conditions on parameters under which the pure strategy N is an ESS.

For given parameter values, the replicator equations are:

$$\begin{cases} \dot{x} = x(2x - 2x^2 - 4xy + 2y^2) \\ \dot{y} = y(4x - 2y - 2x^2 - 4xy + 2y^2) \end{cases} \tag{12}$$

14. From the model equations, determine the parameter values of a, b, c and d of the gain matrix as defined at question 1.

15. Reduce the system at one dimension (variable x).

16. Toward which state does the dynamical process evolve?